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THE COMPLEX WISHART DISTRIBUTION AND THE SYMMETRIC GROUP

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Let V be the space of (r,r) Hermitian matrices and let Ω be the cone of the positive definite ones. We say that the random variable S, taking its values in $\overline{\Omega}$, has the complex Wishart distribution $\gamma_{p,\sigma}$ if $\mathbb{E}(\exp \operatorname{trace}(\theta S)) = (\det(I_r - \sigma\theta))^{-p}$, where σ and $\sigma^{-1} - \theta$ are in Ω, and where $p = 1, 2, \dots, r-1$ or $p > r-1$. In this paper, we compute all moments of S and S^{-1}. The techniques involve in particular the use of the irreducible characters of the symmetric group.

1. The moments of complex Wishart distributions. Let V be the set of (r,r) Hermitian matrices, that is, of the (r,r) matrices s with complex entries such that $s = s^*$, where * indicates the transposed conjugate of the matrix. Note that V is a real linear space, not a complex one. Its dimension is r^2. Let Ω be the open convex cone of the elements s of V which are positive definite, that is, such that $z^* s z > 0$ for any z in $\mathbf{C}^r \setminus \{0\}$. Its closure $\overline{\Omega}$ is the closed convex cone of the elements s of V which are positive, that is, such that $z^* s z \geq 0$ for any z in $\mathbf{C}^r$. Let σ be in Ω and let p be a positive integer. The complex Wishart distribution $\gamma_{p,\sigma}$ on $\overline{\Omega}$ has been considered in the literature [e.g., Goodman (1963)] as the distribution of

$$S = \tfrac{1}{2}(Z_1 Z_1^* + \cdots + Z_p Z_p^*), \tag{1.1}$$

where $Z_1, \dots, Z_p$ are i.i.d. Gaussian centered random variables taking their values in $\mathbf{C}^r$, with covariance such that, if we write $Z_j = X_j + iY_j$ and $\sigma_{jk} = a_{jk} + i b_{jk}$, where $X_j,\ Y_j,\ a_{jk},\ b_{jk}$ are real, then we have

$$\begin{bmatrix} \mathbb{E}(X_j X_k) & \mathbb{E}(X_j Y_k) \\ \mathbb{E}(Y_j X_k) & \mathbb{E}(Y_j Y_k) \end{bmatrix} = \begin{bmatrix} a_{jk} & b_{jk} \\ -b_{jk} & a_{jk} \end{bmatrix}.$$

From (1.1) it is easy to verify that the Laplace transform of S is

$$\mathbb{E}\big(\exp \operatorname{trace}(\theta S)\big) = \big(\det(I_r - \sigma\theta)\big)^{-p}$$

for all θ in V such that $\sigma^{-1} - \theta$ is in Ω. In order to compute this expected value, we use Gaussian integrals and the spectral theorem for Hermitian matrices. From

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its definition in (1.1) and for $p = 1, 2, \dots, r-1$, it is clear that S is concentrated on the boundary $\overline{\Omega} \setminus \Omega$, that is, on the set of singular positive Hermitian matrices. To give a description of the distribution of S for the other values of p, write s in V as $s = (x_{jk} + iy_{jk})_{1\le j,k\le r}$, where x_{jk} and y_{jk} are real and where $y_{jk} = -y_{kj}$, and introduce the Lebesgue measure on V defined by

$$\lambda(ds) = \prod_{j=1}^{r} dx_{jj} \prod_{1\le j<k\le r} dx_{jk}\, dy_{jk}.$$

Suppose now that $p > r-1$, but is not necessarily an integer any more. Then we have

$$\int_{\Omega} (\det s)^{p-r} \exp(-\operatorname{trace} s)\lambda(ds) = \pi^r \Gamma(p)\Gamma(p-1)\cdots\Gamma(p-r+1). \tag{1.2}$$

A proof of (1.2) is essentially in Goodman (1963). The proof given there, for integers $p \ge r$, is easily extended to the interval $(r-1, \infty)$. Another proof can be found in Graczyk, Letac and Massam (2000), which is a more detailed version of the present paper. Denoting by $C(p,r)$ the inverse of the right-hand side of (1.2) we get that

$$C(p,r)\det(\sigma)^{-p}(\det s)^{p-r} \exp(-\operatorname{trace}\sigma^{-1}s)\mathbf{1}_{\Omega}(s)\lambda(ds) \tag{1.3}$$

is a probability. Replacing σ by $(\sigma^{-1}-\theta)^{-1}$ in (1.3), we obtain that for $\sigma^{-1}-\theta$ in Ω,

$$\begin{aligned} C(p,r)&\int_{\Omega} \exp(\operatorname{trace}\theta s)\det(\sigma)^{-p}(\det s)^{p-r} \exp(-\operatorname{trace}\sigma^{-1}s)\mathbf{1}_{\Omega}(s)\lambda(ds) \\ &= \big(\det(I_r - \sigma\theta)\big)^{-p}. \end{aligned}$$

This shows that for $p = r, r+1, \dots$ the probability density function of the $\gamma_{p,\sigma}$ random variable S defined in (1.1) is given explicitly in (1.3). Summarizing the results given above, we say that for $p \in \Lambda = \{1, 2, \dots, r-1\} \cup (r-1, \infty)$ and $\sigma \in \Omega$, the complex Wishart distribution $\gamma_{p,\sigma}$ with shape parameter p and scale parameter σ is defined by its Laplace transform $(\det(I_r - \sigma\theta))^{-p}$. For $p > r-1$ its density is (1.3). For p in the singular part $\{1, 2, \dots, r-1\}$ of Λ, it has the distribution of S as defined in (1.1). The fact that the set of values of p such that $(\det(I_r - \sigma\theta))^{-p}$ is the Laplace transform of a positive measure is equal to Λ, is known as Gyndikin's theorem [see Casalis and Letac (1994) for an accessible proof]. Our definition of $\gamma_{p,\sigma}$ as the complex Wishart distribution differs slightly from the traditional definition given in Goodman (1963) because of the factor $1/2$ in (1.1). The correspondence between the two definitions is $W_r(2p, \frac{\sigma}{2}) = \gamma_{p,\sigma}$.

This paper has been written partly to give a complete answer to the question considered by Maiwald and Kraus (2000) who provide some exact and some approximate formulas for lower moments of the complex Wishart distribution. Our paper extends their results by providing exact formulas for moments of any order, and by trying to give insight into the mechanisms that give us the desired moments.

The practical statistical problem that motivated Maiwald and Kraus (2000) is a common problem in signal processing which can be described as follows. We assume that the data, that is, the signals measured by remote sensors, can be modeled by sequences $(X(t))_{t\in\mathbf{Z}}$ of random variables valued in $\mathbf{C}^r$ and indexed by the set $\mathbf{Z}$ of relative integers, which are stationary centered complex Gaussian processes. Stationarity means that the $r \times r$ complex matrix $C(t,s) = \mathbb{E}(X(t)X(s)^*)$ depends only on $t-s$ [the row vector $X(s)^*$ is the conjugate transpose of $X(s)$]. We are therefore given a sample $X_1,\dots,X_K$ of the initial stationary process X. A complex function $(w(t))_{t\in\mathbf{Z}}$ on $\mathbf{Z}$ such that $\sum_{t\in\mathbf{Z}}|w(t)|^2 = 1$ is called a window. Consider a sequence $(w_T)_0^\infty$ of windows and assume that for all $s \in \mathbf{Z}\setminus\{0\}$, $\lim_{T\to\infty}\sum_{t\in\mathbf{Z}} w_T(t)\overline{w_T(t-s)} = 0$. It can be shown that the limit of the Fourier transform,

$$\tilde{X}(\omega) = \lim_{T\to\infty}\sum_{t\in\mathbf{Z}} w_T(t)X(t)e^{-i\omega t}$$

exists and is such that for $\omega \neq \omega'$, $\tilde{X}(\omega)$ and $\tilde{X}(\omega')$ are independent complex centered random variables [see Brillinger and Krishnaiah (1983), pages 21–28]. The knowledge of the covariance of $\tilde{X}(\omega)$ gives important information on C, the covariance function of X. Indeed under sufficient smoothness conditions the covariance of $\tilde{X}(\omega)$ is the density $f(\omega)$ of the spectral measure of the initial stationary Gaussian process X; that is,

$$C(t,s) = \int_{-\pi}^{\pi} e^{i\omega(t-s)} f(\omega)\,d\omega.$$

An estimate of $f(\omega)$ is given by

$$\hat{f}_K(\omega) = \frac{1}{K}\sum_{k=1}^{K}\tilde{X}_k(\omega)\tilde{X}_k(\omega)^*.$$

This estimate follows, of course, a complex Wishart distribution. We note that $\tilde{X}(\omega)$ is complex even when the initial process X and the windows w_T are real. Thus the complex Wishart distribution, rather than the real one, is the distribution that one has to work with in this type of problem. Some parameters of interest to engineers are complicated functions of the entries of $f(\omega)$ and therefore their corresponding estimates are obtained as functions of the entries of $\hat{f}_K(\omega)$. It is then necessary to obtain an approximation to the distributions of the estimates using the moments of $\hat{f}_K(\omega)$.

In the work that follows we obtain striking formulas that must be used in the type of problem described above. Some of these formulas [Theorem 2, formulas (2.11) and (2.12)] are also helpful in free probability [see Capitaine and Casalis (2002)]. Let us also recall that the moments of the complex Wishart distribution and its inverse can be used in Bayesian inference for problems involving the complex Gaussian distribution.

2. The results. We introduce some notation for the symmetric group $\mathcal{S}_k$ of permutations π of $\{1, 2, \ldots, k\}$.

We recall that a permutation π of a set $\{a_1, \ldots, a_l\}$ of l objects is called a cycle if there exists a φ in $\mathcal{S}_l$ such that for all j one has $\pi(a_{\varphi(j)}) = a_{\varphi(j+1)}$, with the convention $\varphi(l+1) = \varphi(1)$. The permutation π can also be denoted by π_φ. Note that φ is not unique. More specifically, we say that the two bijections φ and φ' are equivalent if there exists an integer q such that $\varphi'(j) = \varphi(j+q)$ for all j, where $j+q$ is taken modulo l. Clearly $\pi_\varphi = \pi_{\varphi'}$ if and only if φ and φ' are equivalent. Note also that if the objects $\{a_1, \ldots, a_l\}$ are (r, r) complex matrices, we use the well-known property of commutativity for traces to see that

$$\operatorname{trace}(a_{\varphi(1)}a_{\varphi(2)}\cdots a_{\varphi(l)}) = \operatorname{trace}(a_{\varphi'(1)}a_{\varphi'(2)}\cdots a_{\varphi'(l)}) \tag{2.1}$$

when φ and φ' are equivalent. When $\pi = \pi_\varphi$ is a cycle, we denote the common value of (2.1) in a symbolic way by

$$\operatorname{trace}\left(\prod_{j\in\pi} a_j\right). \tag{2.2}$$

Now, it is a classical result of group theory that any element π of $\mathcal{S}_k$ can be written in a unique way as the product of cycles built on a set partition of $\{1, 2, \ldots, k\}$. Denote by $C(\pi)$ the set of these cycles, and denote by $m(\pi)$ the size of $C(\pi)$, that is, the number of cycles. For $\pi \in \mathcal{S}_k$, for $\sigma \in \Omega$ and for a sequence $(h_1, \ldots, h_k)$ of (r, r) complex matrices, we now define the quantity

$$r_\pi(\sigma)(h_1, \ldots, h_k) = \prod_{c\in C(\pi)} \operatorname{trace}\left(\prod_{j\in c} \sigma h_j\right). \tag{2.3}$$

Let us illustrate this with an example. Suppose that $k = 6$ and that π is given by $\pi(1) = 6$, $\pi(2) = 5$, $\pi(3) = 1$, $\pi(4) = 4$, $\pi(5) = 2$ and $\pi(6) = 3$. Then $m(\pi) = 3$ and with obvious notation, we write these three cycles as $(1, 6, 3)$, $(2, 5)$ and (4). More commonly, we write $\pi = (1, 6, 3)(2, 5)(4)$. The order on the set $C(\pi)$ and the circular order inside the cycles are irrelevant and we can also write $\pi = (4)(2, 5)(6, 3, 1)$ for instance. For the particular example above, $r_\pi(\sigma)(h_1, \ldots, h_6) = \operatorname{trace}(\sigma h_4)\operatorname{trace}(\sigma h_2\sigma h_5)\operatorname{trace}(\sigma h_1\sigma h_6\sigma h_3)$.

The two main results of this paper are given in Theorem 2 and Theorem 4 below, which link arbitrary moments and inverse moments of the complex Wishart distribution, respectively, with the symmetric group $\mathcal{S}_k$. Theorem 1 is a particular case of Theorem 2, but is actually a crucial ingredient in its proof and deserves a separate treatment.

THEOREM 1. *Let V be the space of (r, r) Hermitian matrices, let σ be in the cone Ω of positive definite elements of V and let p be in the Gyndikin set $\Lambda = \{1, 2, \ldots, r-1\} \cup (r-1, \infty)$. Consider a random variable S in V with the* *complex Wishart distribution $\gamma_{p,\sigma}$ and let $(h_1,\dots,h_k)$ be arbitrary (r,r) complex matrices. Then*

$$\mathbb{E}\big(\operatorname{trace}(Sh_1)\operatorname{trace}(Sh_2)\cdots\operatorname{trace}(Sh_k)\big)=\sum_{\pi\in\mathcal{S}_k}p^{m(\pi)}r_\pi(\sigma)(h_1,\dots,h_k). \tag{2.4}$$

Theorem 1 is of interest in its own right because it yields immediately the simplest moments of $\gamma_{p,\sigma}$. Let $S=(S_{jl})_{1\le j,l\le r}=(X_{jl}+iY_{jl})_{1\le j,l\le r}$ be an element of Ω and suppose that we want to calculate the real number

$$\mathbb{E}\left(\prod_{1\le j,l\le r}X_{jl}^{a_{jl}}Y_{jl}^{b_{jl}}\right) \tag{2.5}$$

for some integers a_{jl} and b_{jl}. One can verify that if $E_{k,N}$ is the complex space of homogeneous polynomials of degree N with respect to the $2k$ variables $x_1,\dots,x_k,y_1,\dots,y_k$, and if $I_{k,N}$ is the set of sequences of $2k$ nonnegative integers $a_1,\dots,a_k,b_1,\dots,b_k$ such that $a_1+\cdots+a_k+b_1+\cdots+b_k=N$, then the two sets of polynomials,

$$\{x_1^{a_1}\cdots x_k^{a_k}y_1^{b_1}\cdots y_k^{b_k};\ (a_1,\dots,a_k,b_1,\dots,b_k)\in I_{k,N}\}$$

and

$$\begin{aligned}\{(x_1+iy_1)^{a_1}\cdots(x_k+iy_k)^{a_k}(x_1-iy_1)^{b_1}\cdots(x_k-iy_k)^{b_k};\\ (a_1,\dots,a_k,b_1,\dots,b_k)\in I_{k,N}\},\end{aligned}$$

both form a basis of $E_{k,N}$. Thus, the elements of the first basis can be expressed as linear combinations of the elements of the second one. As a consequence, in order to calculate (2.5), it is enough to compute complex moments of the form

$$\mathbb{E}\left(\prod_{1\le j,l\le r}S_{jl}^{a_{jl}}\overline{S}_{jl}^{b_{jl}}\right). \tag{2.6}$$

This can be done by applying Theorem 1 to suitable matrices $h_1,\dots,h_k$ [see (4.2)].

The statement of Theorem 2 includes in particular all the formulas of Section 4 of Maiwald and Kraus (2000). Indeed, for $k=1$ and $k=2$, (2.7) gives the same results as their formulas (15)–(17). For $k=3$, their formula (18) gives the coefficient of p^3 and p^2 in (2.7) for $\pi=(1,2)(3)$. For $k=4$, their formula (19) gives the coefficient of p^4 and p^3 in (2.7) for $\pi=(1,2)(3,4)$.

THEOREM 2. *With the notation of Theorem 1, for all $\pi\in\mathcal{S}_k$ we have*

$$\mathbb{E}\big(r_\pi(S)(h_1,\dots,h_k)\big)=\sum_{\pi'\in\mathcal{S}_k}p^{m(\pi'^{-1}\circ\pi)}r_{\pi'}(\sigma)(h_1,\dots,h_k). \tag{2.7}$$

Our next result, Theorem 3, links $\mathbb{E}(r_\pi(S^{-1}))$ and $r_\pi(\sigma^{-1})$ and is a first step towards the proof of Theorem 4, which is the main result for the moments of the inverse complex Wishart distribution.

THEOREM 3. *With the notation of Theorem* 1, *writing* $q = p - r$ *and assuming that* $q > k$, *we have, for all* $\pi \in \mathcal{S}_k$,

$$\begin{aligned} &r_\pi(\sigma^{-1})(h_1, \dots, h_k) \\ &\quad = (-1)^k \sum_{\pi' \in \mathcal{S}_k} (-q)^{m(\pi'^{-1} \circ \pi)} \mathbb{E}\big(r_{\pi'}(S^{-1})(h_1, \dots, h_k)\big). \end{aligned} \tag{2.8}$$

In the statement of the last theorem, Theorem 4, we use the characters of the symmetric group $\mathcal{S}_k$. For the convenience of the reader, this concept will be recalled in Section 6. [Our reference is Simon (1996).] To state this theorem, let us introduce some notation.

We denote by I_k the set of sequences $\mathbf{i} = (i_1, \dots, i_k)$ of k nonnegative integers such that $i_1 + 2i_2 + \cdots + ki_k = k$. To each π in $\mathcal{S}_k$ we associate the element $\mathbf{i} = \mathbf{i}(\pi) = (i_1, \dots, i_k)$ of I_k where i_j is the number of cycles of length j in π. Such an element $\mathbf{i}$ of I_k is called the *portrait* of π. For instance $(k, 0, \dots, 0)$ is the portrait of the identity and is abbreviated as (k). It is easy to see [Simon (1996), Theorem VI 1.2] that the size $\sharp\mathbf{i}$ of the set of π's with portrait $\mathbf{i}$ is

$$\sharp\mathbf{i} = \frac{k!}{\prod_{j=1}^k i_j! j^{i_j}}. \tag{2.9}$$

If a function on $\mathcal{S}_k$, say, $\pi \mapsto a(\pi)$, depends only on the portrait $\mathbf{i}$ of π, we write $a(\mathbf{i}) = a(\pi)$, that is a is considered as a function on I_k.

We introduce also the set M_k of sequences $\mathbf{m} = (m_1, \dots, m_k)$ of integers such that $m_1 \ge m_2 \ge \cdots \ge m_k \ge 0$ and such that $m_1 + \cdots + m_k = k$.

Finally, we consider the matrix

$$C_k = \big(\chi^{\mathbf{m}}(\mathbf{i})\big)_{\mathbf{m} \in M_k,\ \mathbf{i} \in I_k}$$

of characters $\chi^{\mathbf{m}}(\mathbf{i})$ of $\mathcal{S}_k$, and we denote by $d_{\mathbf{m}}$ the dimension of the irreducible representation associated with $\mathbf{m}$ in M_k; that is [Simon (1996), Theorem VI.2.3, page 98],

$$d_{\mathbf{m}} = \frac{k!}{\prod_{i=1}^k (m_i - i + k)} \prod_{1 \le i < j \le k} (m_i - m_j - i + j). \tag{2.10}$$

THEOREM 4. *We keep the same assumptions and notation as in Theorem* 3.

(i) *For* $\mathbf{m} \in M_k$ *we denote*

$$f_{\mathbf{m}} = \frac{d_{\mathbf{m}}}{k!} \prod_{j=1}^{k} \prod_{i=1}^{m_j} (q + j - i). \tag{2.11}$$

Then for all π *in* $\mathcal{S}_k$ *we have*

$$(-1)^k(-q)^{m(\pi)} = \sum_{\mathbf{m}\in M_k} f_{\mathbf{m}}\chi^{\mathbf{m}}(\mathbf{i}). \tag{2.12}$$

(ii) *For* π *in* $\mathcal{S}_k$ *we denote*

$$f^{(-1)}(\pi) = \sum_{\mathbf{m}\in M_k} \frac{d_{\mathbf{m}}^2}{k!^2 f_{\mathbf{m}}}\chi^{\mathbf{m}}(\pi). \tag{2.13}$$

Then the inverse formula of (2.8) *is*

$$\mathbb{E}\big(r_\pi(S^{-1})(h_1,\dots,h_k)\big) = \sum_{\pi'\in\mathcal{S}_k} f^{(-1)}(\pi'^{-1}\circ\pi) r_{\pi'}(\sigma^{-1})(h_1,\dots,h_k). \tag{2.14}$$

In particular for $\pi =$ *identity we obtain*

$$\begin{aligned}&\mathbb{E}\big(\operatorname{trace}(S^{-1}h_1)\operatorname{trace}(S^{-1}h_2)\cdots\operatorname{trace}(S^{-1}h_k)\big)\\ &\quad= \sum_{\pi'\in\mathcal{S}_k} f^{(-1)}(\pi')r_{\pi'}(\sigma^{-1})(h_1,\dots,h_k).\end{aligned} \tag{2.15}$$

Note that we can use (2.15) in Theorem 4 to compute moments of S^{-1} of the form $\mathbb{E}(\prod_{1\le j,l\le r} S_{jl}^{-a_{jl}}\overline{S}_{jl}^{-b_{jl}})$, just as we used Theorem 1 to compute similar moments of S. For $k\le 4$, Theorem 4 gives as a particular case formulas 50–54 of Maiwald and Kraus (2000).

To conclude this list of results, we observe that, for $k=1$ and $k=2$, the previous theorems give results which have already appeared in the literature in the more general context of Wishart distributions on Jordan algebras. Indeed the present space V of (r,r) Hermitian matrices can be considered as a Jordan algebra with Peirce constant $d=2$. In particular, the following formulas,

$$\mathbb{E}(S) = p\sigma,$$
$$\mathbb{E}(S^{-1}) = \sigma^{-1}/q$$

and

$$\begin{aligned}\mathbb{E}\big(\operatorname{trace}(Sh_1)\operatorname{trace}(Sh_2)\big) &= p^2\operatorname{trace}(\sigma h_1)\operatorname{trace}(\sigma h_2) + p\operatorname{trace}(\sigma h_1\sigma h_2),\\ \mathbb{E}\big(\operatorname{trace}(Sh_1Sh_2)\big) &= p^2\operatorname{trace}(\sigma h_1\sigma h_2) + p\operatorname{trace}(\sigma h_1)\operatorname{trace}(\sigma h_2),\\ \mathbb{E}\big(\operatorname{trace}(S^{-1}h_1)\operatorname{trace}(S^{-1}h_2)\big) &= \frac{1}{q^3-q}\big(q\operatorname{trace}(\sigma^{-1}h_1)\operatorname{trace}(\sigma^{-1}h_2)\\ &\qquad + \operatorname{trace}(\sigma^{-1}h_1\sigma^{-1}h_2)\big),\\ \mathbb{E}\big(\operatorname{trace}(S^{-1}h_1S^{-1}h_2)\big) &= \frac{1}{q^3-q}\big(\operatorname{trace}(\sigma^{-1}h_1\sigma^{-1}h_2)\\ &\qquad + q\operatorname{trace}(\sigma^{-1}h_1)\operatorname{trace}(\sigma^{-1}h_2)\big),\end{aligned}$$

have already appeared in Letac and Massam (1998), formula (6.1), and in Letac and Massam (2000), formulas (5.1), (6.1) and (6.2). Graczyk, Letac and Massam (2000) put the results of the present paper in the proper perspective of Jordan algebras.

We also note that for the real case, moments of Wishart and inverse Wishart random variables have been investigated by different authors. Let us mention here von Rosen (1988) and several subsequent papers, as well as Wong and Liu (1995). Their methods could possibly be imitated and provide results for the complex Wishart. The intricacy of the real case makes even more surprising the simplicity of the results given by Maiwald and Kraus (2000) and their extension as given in Theorems 2 and 3 of this paper.

3. Proof of Theorem 1. We observe first that for $p \in \Lambda$, there exists an unbounded measure μ_p on $\overline{\Omega}$ such that for all θ in $-\Omega$ the Laplace transform of μ_p is

$$L_{\mu_p}(\theta) = \int_{\overline{\Omega}} e^{\operatorname{trace}(\theta s)} \mu_p(ds) = (\det(-\theta))^{-p} = \big(\det(-\theta)^{-1}\big)^p. \tag{3.1}$$

The definition of $\gamma_{p,\sigma}$ through its Laplace transform as we have given it in Section 1 shows that $\gamma_{p,\sigma}(ds) = (\det\sigma)^{-p}\exp(-\operatorname{trace}(\sigma^{-1}s))\mu_p(ds)$. In other words, for a fixed p, $\{\gamma_{p,\sigma};\ \sigma \in \Omega\}$ is the natural exponential family generated by μ_p. For this reason, it is useful to use the natural parametrization of this family given by $\theta \in -\Omega$. The more traditional parameter $\sigma \in \Omega$ is such that

$$\sigma = \sigma(\theta) = (-\theta)^{-1}. \tag{3.2}$$

Let κ be the function defined on $-\Omega$ by $\kappa(\theta) = \log\det(-\theta)^{-1} = \log\det\sigma$, so that $L_{\mu_p} = \exp(p\kappa)$. We will use in the sequel the following two rules of differentiation of κ and σ with respect to θ in the direction $h \in V$:

$$\sigma'(\theta)(h) = \sigma h \sigma, \qquad \kappa'(\theta)(h) = \operatorname{trace}(\sigma h). \tag{3.3}$$

These formulas follow immediately from the rules of differentiation of a determinant and of the inverse of a matrix. Formula (3.3) provides a remarkable way to differentiate the function $\theta \mapsto \operatorname{trace}(\sigma h_1 \sigma h_2 \cdots \sigma h_k)$ in the direction h_{k+1}. The differential of that function in the direction h_{k+1} is equal to

$$\operatorname{trace}(\sigma h_{k+1}\sigma h_1 \cdots \sigma h_k) + \operatorname{trace}(\sigma h_1 \sigma h_{k+1} \sigma h_2 \cdots \sigma h_k) \\ + \cdots + \operatorname{trace}(\sigma h_1 \sigma h_2 \cdots \sigma h_{k+1} \sigma h_k).$$

It has been obtained by substituting $\sigma h_{k+1} \sigma$ for σ, k times in $\theta \mapsto \operatorname{trace}(\sigma h_1 \sigma h_2 \cdots \sigma h_k)$.

More generally, consider some cycle c of $\pi \in \mathcal{S}_k$, say, $c = (\varphi_1, \dots, \varphi_l)$. If $\nu = \varphi_i$, we write c_ν for the cycle $(\varphi_1, \dots, \varphi_{i-1}, k+1, \varphi_i, \dots, \varphi_l)$ of $\mathcal{S}_{k+1}$. Then the differential of $\theta \mapsto \operatorname{trace}\prod_{j \in c}(\sigma h_j)$ taken in the direction h_{k+1} is

$$\sum_{\nu \in c} \operatorname{trace} \prod_{j \in c_\nu} (\sigma h_j). \tag{3.4}$$

The heart of the proof is the following algebraic lemma.

LEMMA 5. *Let* $\sigma(\theta) = (-\theta)^{-1}$, *with* $\det\theta \neq 0$. *Then the kth differential form of* $\theta \mapsto (\det\sigma)^p$ *is the multilinear form*

$$(h_1, \dots, h_k) \mapsto (\det\sigma)^p \left(\sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(h_1, \dots, h_k) \right). \tag{3.5}$$

PROOF. Using (3.3), the result is clear for $k = 1$. Let us now assume that it is true for k. Let us take the differential of (3.5) with respect to θ in the direction h_{k+1}. Using (3.3) and (3.4), we obtain

$$\begin{aligned} &p(\det\sigma)^p \operatorname{trace}(\sigma h_{k+1}) \left(\sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(h_1, \dots, h_k) \right) \\ &\quad + (\det\sigma)^p \left(\sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} \sum_{c \in C(\pi)} \left(\sum_{\nu \in c} \operatorname{trace} \prod_{j \in c_\nu} (\sigma h_j) \right) \right) \\ &\quad \times \left(\prod_{c_1 \in C(\pi) \setminus \{c\}} \operatorname{trace} \prod_{l \in c_1} h_l \right). \end{aligned}$$

We now split $\mathcal{S}_{k+1}$ into two parts, the first part containing the π's in $\mathcal{S}_{k+1}$ such that $\pi(k+1) = k+1$, and the second part containing the remaining permutations. Clearly, the two preceding lines correspond to each one of the two parts and it follows that the $(k+1)$th differential of $\theta \mapsto (\det\sigma)^p$ is

$$(h_1, \dots, h_{k+1}) \mapsto (\det\sigma)^p \left(\sum_{\pi \in \mathcal{S}_{k+1}} p^{m(\pi)} r_\pi(\sigma)(h_1, \dots, h_{k+1}) \right).$$

The induction argument is therefore completed and the lemma is proved. $\square$

Let us now prove Theorem 1. We will do so first for $(h_1, \dots, h_k)$ Hermitian. If S is $\gamma_{p,\sigma}$ distributed, taking the kth differential of (3.1) in the directions $(h_1, \dots, h_k)$ yields immediately

$$\mathbb{E}\big(\operatorname{trace}(Sh_1) \operatorname{trace}(Sh_2) \cdots \operatorname{trace}(Sh_k)\big) = \frac{1}{L_{\mu_p}(\theta)} L^{(k)}_{\mu_p}(\theta)(h_1, \dots, h_k),$$

which can be rewritten

$$L^{(k)}_{\mu_p}(\theta)(h_1, \dots, h_k) = (\det\sigma)^p \mathbb{E}\big(\operatorname{trace}(Sh_1) \operatorname{trace}(Sh_2) \cdots \operatorname{trace}(Sh_k)\big). \tag{3.6}$$

Putting together (3.1), Lemma 5 and (3.6) gives the proof of (2.4) for Hermitian matrices h_j.

Let us now prove (2.4) for matrices which are not necessarily Hermitian. Consider the Hermitian matrices $a_j(0) = \frac{1}{2}(h_j + h_j^*)$ and $a_j(1) = \frac{1}{2i}(h_j - h_j^*)$, so that $h_j = a_j(0) + ia_j(1)$. Consider also all sequences $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in \{0, 1\}^k$. We write $|\varepsilon| = \varepsilon_1 + \cdots + \varepsilon_k$.

Using (2.4) for Hermitian matrices and the linearity of traces we have

$$\begin{aligned}
&\mathbb{E}\left(\prod_{j=1}^{k} \operatorname{trace}\big(S\big(a_j(0) + ia_j(1)\big)\big)\right)\\
&\quad = \sum_{\varepsilon\in\{0,1\}^k} i^{|\varepsilon|}\mathbb{E}\left(\prod_{j=1}^{k} \operatorname{trace}\big(Sa_j(\varepsilon_j)\big)\right)\\
&\quad = \sum_{\varepsilon\in\{0,1\}^k} i^{|\varepsilon|}\left(\sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)\big(a_1(\varepsilon_1), \dots, a_k(\varepsilon_k)\big)\right)\\
&\quad = \sum_{\varepsilon\in\{0,1\}^k} \left(\sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)\big(i^{\varepsilon_1}a_1(\varepsilon_1), \dots, i^{\varepsilon_k}a_k(\varepsilon_k)\big)\right)\\
&\quad = \sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)\big(a_1(0) + ia_1(1), \dots, a_k(0) + ia_k(1)\big),
\end{aligned}$$

which proves (2.4) for arbitrary complex matrices.

4. Proof of Theorem 2. We begin with a corollary to Theorem 1. We first need to introduce some notation. The set $\{1, 2, \dots, r\}^{2k}$ of double sequences $B = (a_1, b_1, a_2, b_2, \dots, a_k, b_k)$ of integers in $\{1, 2, \dots, r\}$ is denoted by D_k. For π in the symmetric group $\mathcal{S}_k$, we define the transformation l_π on D_k by

$$l_\pi(B) = (a_1, b_{\pi(1)}, a_2, b_{\pi(2)}, \dots, a_k, b_{\pi(k)}).$$

Thus $(\pi, B) \mapsto l_\pi(B)$ is a group action of $\mathcal{S}_k$ on D_k, and $\pi \mapsto l_\pi$ is an injective antihomomorphism (that is to say, $l_{\pi'\circ\pi} = l_\pi \circ l_{\pi'}$) between $\mathcal{S}_k$ and the group of permutations of D_k.

Let us also adopt the following notation: if $F = (F_{ab})_{1\le a,b\le r}$ is an (r, r) matrix and if $B = (a_1, b_1, a_2, b_2, \dots, a_k, b_k)$ is in D_k, we write

$$F(B) = F_{a_1b_1}F_{a_2b_2}\cdots F_{a_kb_k}.$$

Since $l_\pi(B)$ is in D_k, $F(l_\pi(B))$ is also well defined. Suppose for instance that $k = 6$ and that $\pi = (1, 6, 3)(2, 5)(4)$. Then

$$F(l_\pi(B)) = F_{a_1b_6}F_{a_6b_3}F_{a_3b_1}F_{a_2b_5}F_{a_5b_2}F_{a_4b_4}.$$

We are now in position to state the corollary of Theorem 1. It extends the formulas given between the formulas (14) and (15) in Maiwald and Kraus (2000) for $k = 2, 3, 4$.

COROLLARY 6. *For all $B \in D_k$ we have*

$$\mathbb{E}(S(B)) = \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} \sigma(l_\pi(B)). \tag{4.1}$$

PROOF. We introduce here the following notation which will also be useful later: for $(a, b) \in \{1, 2, \dots, r\}^2$, let

$$h^{ab} = (h^{ab}_{ts})_{1 \le t, s \le r} \tag{4.2}$$

be the (r, r) matrix with $h^{ab}_{ba} = 1$ and $h^{ab}_{ts} = 0$ if $(t, s) \neq (b, a)$. The matrix h^{ab} is designed so that trace $Sh^{ab} = S_{ab}$.

We now apply Theorem 1 to $h_j = h^{a_j b_j}$, where $B = (a_1, b_1, a_2, b_2, \dots, a_k, b_k)$. Since trace $Sh^{ab} = S_{ab}$, it is clear that the first member of (2.4) is equal to $\mathbb{E}(S(B))$. To compute the second member of (2.4), we observe that if $(A^{(1)}, \dots, A^{(n)})$ is a sequence of (r, r) matrices, then it can easily be proved by induction on n that

$$\operatorname{trace}(A^{(1)} A^{(2)} \cdots A^{(n)}) = \sum_{1 \le j_1, \dots, j_n \le r} A^{(1)}_{j_1 j_2} A^{(2)}_{j_2 j_3} \cdots A^{(n-1)}_{j_{n-1} j_n} A^{(n)}_{j_n j_1}.$$

Applying this formula to $A^{(1)} = F$, $A^{(2)} = h^{a_1 b_1}$, $A^{(3)} = F$, etc., we get that for any (r, r) matrix F,

$$\operatorname{trace}\bigl(F h^{a_1 b_1} F h^{a_2 b_2} \cdots F h^{a_k b_k}\bigr) = F_{a_1 b_2} \cdots F_{a_{k-1} b_k} F_{a_k b_1}. \tag{4.3}$$

We notice that the second member of (4.3) is $F(l_\pi(B))$ when π is the cycle $(1, 2, \dots, k)$. Applying this to each cycle of $\pi \in \mathcal{S}_k$ and using the definition of r_π given in (2.3) we obtain

$$r_\pi(F)\bigl(h^{a_1 b_1}, \dots, h^{a_k b_k}\bigr) = F(l_\pi(B)). \tag{4.4}$$

Letting $F = \sigma$ and summing both sides of (4.4) over all $\pi \in \mathcal{S}_k$ yields (4.1) and the corollary is proved. □

We are now in position to prove Theorem 2. For $B = (a_1, b_1, a_2, b_2, \dots, a_k, b_k)$ in D_k given, we will prove Theorem 2 first for $h_j = h^{a_j b_j}$. For these h_j, it follows immediately from (4.4) applied to $F = S$ that the left-hand side of (2.7) is $\mathbb{E}(S(l_\pi(B)))$ while the right-hand side of (2.7) is

$$\sum_{\pi' \in \mathcal{S}_k} p^{m(\pi^{-1} \circ \pi')} \sigma(l_{\pi'}(B)).$$

[Recall that $m(\pi) = m(\pi^{-1})$.] Making the change of variable $\pi'' = \pi^{-1} \circ \pi'$ in this last sum we obtain

$$\sum_{\pi'' \in \mathcal{S}_k} p^{m(\pi'')} \sigma(l_{\pi \circ \pi''}(B)).$$

Now, we use the fact that $\pi \mapsto l_\pi$ is an antihomomorphism. Applying (4.1) to $B' = l_\pi(B)$ yields (2.7) for $h_j = h^{a_j b_j}$.

In order to obtain (2.7) for any sequence $(h_1, \dots, h_k)$ of (r, r) complex matrices, we use linearity in a standard way and the proof of Theorem 2 is complete.

5. Proof of Theorem 3. Similarly to what we did for Theorem 2, we first prove Theorem 3 for $\pi =$ identity and for Hermitian matrices h_j. The idea of the proof is to use the Stokes formula, roughly in the following way: If a function f defined on $\overline{\Omega}$ is zero on the boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$ and is smooth enough, then $\int_\Omega f'(s)(h)\lambda(ds) = 0$. Actually, we use this principle for the kth differential of f, obtaining

$$\int_\Omega f^{(k)}(s)(h_1, \dots, h_k)\lambda(ds) = 0. \tag{5.1}$$

The formula is applied to $s \mapsto f(s) = C(p,r)(\det s)^q \exp(\mathrm{trace}(\theta s))$, where $C(p,r)$ is the constant in (1.3) and $\theta = -\sigma^{-1}$ is in $-\Omega$. Since f is the product of the two functions $u(s) = (\det s)^q$ and $v(s) = C(p,r)\exp(\mathrm{trace}(\theta s))$, we first apply Leibnitz's formula,

$$(uv)^{(k)}(s)(h_1, \dots, h_k) = \sum_{T \subset \{1,\dots,k\}} u^{(T)}(s)(h_T)v^{(T')}(s)(h_{T'}), \tag{5.2}$$

with the following notation: $T' = \{1, \dots, k\} \setminus T$ is the complementary set of T, h_T is the set of variables $(h_j)_{j\in T}$ and $u^{(T)}(s)$ means the differential whose order is the size of T. For the particular choices that we have made for u and v, these differentials are easy to compute. Indeed, the differentials of v are such that

$$\begin{aligned} v^{(k)}(s)(h_1, \dots, h_k) &= v(s) \times \mathrm{trace}(\theta h_1) \cdots \mathrm{trace}(\theta h_k) \\ &= v(s) \times (-1)^k\, \mathrm{trace}(\sigma^{-1}h_1) \cdots \mathrm{trace}(\sigma^{-1}h_k). \end{aligned}$$

To compute the differentials of u we use Lemma 5 in Section 3, with θ replaced by $-s$ so that $(\det s)^q = (\det(-\theta)^{-1})^{-q}$, and with p replaced by $-q$. We obtain

$$u^{(k)}(s)(h_1, \dots, h_k) = u(s) \times (-1)^k \sum_{\pi \in \mathcal{S}_k} (-q)^{m(\pi)} r_\pi(s^{-1})(h_1, \dots, h_k). \tag{5.3}$$

For simplicity, denote $v_j = \mathrm{trace}(\sigma^{-1}h_j)$ and for $T \subset \{1, \dots, k\}$,

$$u(T) = \int_\Omega u^{(T)}(s)(h_T)v(s)\lambda(ds).$$

Formula (5.3) implies in particular that

$$u(\{1, \dots, k\}) = (-1)^k \sum_{\pi \in \mathcal{S}_k} (-q)^{m(\pi)} \mathbb{E}\big(r_\pi(S^{-1})(h_1, \dots, h_k)\big). \tag{5.4}$$

Note that this implies that $u(\varnothing) = 1$. Using (5.1), (5.2) and the computations of the differentials of u and v we have

$$\sum_{T \subset \{1,\dots,k\}} u(T) \prod_{j \in T'} (-v_j) = 0.$$

More generally, from the last formula, we get that, for all nonempty $T_1 \subset \{1, \dots, k\}$,

$$\sum_{T \subset T_1} u(T) \prod_{j \in T_1 \setminus T} (-v_j) = 0.$$

It is easily seen by induction on the size of T_1 that the solution of this linear system is exactly $u(T_1) = \prod_{j \in T_1} v_j$, since $u(\varnothing) = 1$. In particular,

$$u(\{1, 2, \dots, k\}) = \prod_{j=1}^{k} \operatorname{trace}(\sigma^{-1} h_j)$$

and therefore (5.4) yields the desired formula (2.8) of Theorem 3 for $\pi =$ identity and Hermitian matrices h_j.

The remainder of the proof of Theorem 3 follows the same lines as the proofs of the two previous theorems; that is, as in Theorem 1, in a first step, we prove (2.8) for $\pi =$ identity and for arbitrary h_j, by a linearity argument. In a second step, keeping $\pi =$ identity and choosing h_j to be $h_j = h^{a_j b_j}$, as defined in (4.2), (2.8) from the first step becomes

$$\sigma^{-1}(B) = (-1)^k \sum_{\pi \in \mathcal{S}_k} (-q)^{m(\pi)} \mathbb{E}\big(S^{-1}(l_\pi(B))\big),$$

where $B = (a_1, b_1, \dots, a_k, b_k)$. In the third step, we apply the formula above to $B' = l_\pi(B)$ in order to obtain (2.8) for the particular $h_j = h^{a_j b_j}$ but for arbitrary π. Finally a linearity argument leads us to the proof of (2.8) in Theorem 3.

6. The case $k = 3$ and the irreducible representations of $\mathcal{S}_k$. The remainder of the paper is devoted to the computation of the $\mathbb{E}(r_\pi(S^{-1})(h_1, \dots, h_k))$, that is, to the inversion of the $(k!, k!)$ matrix $M(q) = ((-1)^k(-q)^{m(\pi'^{-1} \circ \pi)})$ appearing in (2.8) and to the proof of Theorem 4. In this section, we use the case $k = 3$ to show that methods from group theory arise naturally in this problem.

Let us denote the elements of $\mathcal{S}_3$ by

$$\pi_1 = (1)(2)(3), \quad \pi_2 = (1, 2, 3), \quad \pi_3 = (1, 3, 2),$$
$$\pi_4 = (1)(2, 3), \quad \pi_5 = (2)(1, 3), \quad \pi_6 = (3)(1, 2).$$

Following the order given by the π_i, and denoting by i_3 and j_3 the identity (3, 3) matrix and the matrix containing only ones, respectively, we can write the matrix $M(q)$ by blocks as

$$M(q) = \begin{bmatrix} (q^3 - q)i_3 + qj_3 & -q^2 j_3 \\ -q^2 j_3 & (q^3 - q)i_3 + qj_3 \end{bmatrix},$$

and therefore the matrix $(M(q))^{-1}$ is easily calculated:

$$(6.1)\quad (M(q))^{-1} = \frac{1}{q(q^2-1)(q^2-4)} \begin{bmatrix} (q^2-4)i_3 + 2j_3 & qj_3 \\ qj_3 & (q^2-4)i_3 + 2j_3 \end{bmatrix}.$$

Denoting $P(q) = q(q^2-1)(q^2-4)$, this leads to the following explicit solution of the moment problem for S^{-1} and for $k = 3$:

$$\begin{bmatrix} \mathbb{E}(r_{\pi_1}(S^{-1})) \\ \mathbb{E}(r_{\pi_2}(S^{-1})) \\ \mathbb{E}(r_{\pi_3}(S^{-1})) \\ \mathbb{E}(r_{\pi_4}(S^{-1})) \\ \mathbb{E}(r_{\pi_5}(S^{-1})) \\ \mathbb{E}(r_{\pi_6}(S^{-1})) \end{bmatrix}$$

$$= \frac{1}{P(q)} \begin{bmatrix} q^2-2 & 2 & 2 & q & q & q \\ 2 & q^2-2 & 2 & q & q & q \\ 2 & 2 & q^2-2 & q & q & q \\ q & q & q & q^2-2 & 2 & 2 \\ q & q & q & 2 & q^2-2 & 2 \\ q & q & q & 2 & 2 & q^2-2 \end{bmatrix} \begin{bmatrix} r_{\pi_1}(\sigma^{-1}) \\ r_{\pi_2}(\sigma^{-1}) \\ r_{\pi_3}(\sigma^{-1}) \\ r_{\pi_4}(\sigma^{-1}) \\ r_{\pi_5}(\sigma^{-1}) \\ r_{\pi_6}(\sigma^{-1}) \end{bmatrix}.$$

It is also interesting to compute the eigenspaces and eigenvalues of $M(q)$. We obtain the eigenvalue $q^3 - 3q^2 + 2q$, associated with the eigenvector $[1, 1, 1, 1, 1, 1]$, the eigenvalue $q^3 + 3q^2 + 2q$ associated with the eigenvector $[1, 1, 1, -1, -1, -1]$ and finally $q^3 - q$, with multiplicity 4, associated with the four-dimensional space orthogonal to the two preceding eigenvectors. Note that these eigenspaces do not depend on q.

To compute $\mathbb{E}(r_\pi(S^{-1})(h_1, \dots, h_k))$ for higher values of k, we will have to use methods which are less elementary than before, and we will have to rely on classical results for finite groups, which we will apply later to the group $G = \mathcal{S}_k$.

For the reader who is not interested in mathematical details, it is sufficient to say that for $k \geq 3$, the matrix equivalent to $M(q)$ is the matrix representative of a linear application from $\mathbf{C}^{\mathcal{S}_k}$ to itself. To find the expression of the $\mathbb{E}(r_\pi(\sigma^{-1}))$ we need to compute the inverse of this linear application. This will be done using some characteristics of the group $\mathcal{S}_k$, in particular, its characters.

For the reader who prefers a deeper comprehension of the proof of Theorem 4, we now give the general mathematical ideas used in our proof. Given a finite group G, its algebra $\mathcal{A}(G)$ is the complex linear space of functions from G to $\mathbf{C}$. This space is endowed with a product operation called *convolution* and defined by

$$f * g(x) = \sum_{y \in G} f(y^{-1}x) g(y).$$

One can easily verify that this product on $\mathcal{A}(G)$ is associative. Its unit is the indicator δ_e of the neutral element e of G. We give $\mathcal{A}(G)$ a Hermitian structure by defining the scalar product

$$\langle f, g\rangle = \frac{1}{o(G)} \sum_{x\in G} \overline{f(x)} g(x).$$

On G, we say that x is equivalent to x' if and only if there exists y in G such that $x' = yxy^{-1}$. The equivalence classes of this relation are simply called *classes*. The subspace $\mathcal{Z}(G)$ of $\mathcal{A}(G)$ of the functions f such that f is constant on each class is called the *center* of $\mathcal{A}(G)$, and an element of $\mathcal{Z}(G)$ is called a *central* function. The reason for this name is that $f * g = g * f$ for all g in $\mathcal{A}(G)$ if and only if f is central [see Simon (1996), page 39].

For a given central function f, we now want to analyse the endomorphism $L(f)$ of $\mathcal{A}(G)$ defined by $g \mapsto f * g = L(f)(g)$, by finding in particular its eigenspaces and eigenvalues. To do so, we introduce the set $\hat{G}$ of all irreducible linear representations of G (up to equivalence). Consider also the regular linear representation defined as the family of unitary transformations $(R(y))_{y\in G}$ of the Hermitian space $\mathcal{A}(G)$ acting as follows: for all g in $\mathcal{A}(G)$, then $R(y)g(x) = g(xy)$. From the theory of finite groups, we know that R is the direct sum of all irreducible representations $D^{(\alpha)}$, that is, each $\alpha \in \hat{G}$ is present in the canonical decomposition of R with multiplicity d_α, where d_α denotes the dimension of the representation $D^{(\alpha)}$ [see Simon (1996), Section III.1, or more explicitly Fulton and Harris (1991), Corollary 2.18]. Without loss of generality, let us choose a representative unitary matrix $D^{(\alpha)}(x) = (D_{ij}^{(\alpha)}(x))_{1\le i,j\le d_\alpha}$.

Then $\mathcal{A}(G)$ is the orthogonal direct sum of subspaces V_α of respective dimensions d_α^2 which are generated by the set of functions on G which are the coefficients of the representation, namely $\{D_{ij}^{(\alpha)}(x);\ 1 \le i,\ j \le d_\alpha\}$. Furthermore the $D_{ij}^{(\alpha)}$ are orthogonal, with $\langle D_{ij}^{(\alpha)}, D_{ij}^{(\alpha)}\rangle = 1/d_\alpha$ [see Simon (1996), page 36]. Each V_α shares exactly a one-dimensional linear space with $\mathcal{Z}(G)$, and this space is generated by the *character*,

$$\chi^{(\alpha)}(x) = \operatorname{trace} D^{(\alpha)}(x) = \sum_{i=1}^{d_\alpha} D_{ii}^{(\alpha)}(x),$$

of the corresponding irreducible representation. These characters satisfy $\langle \chi^{(\alpha)}, \chi^{(\alpha)}\rangle = 1$. Since the $(V_\alpha)_{\alpha\in\hat{G}}$ form an orthogonal direct sum, the characters are orthonormal. In fact, the characters form a basis of $\mathcal{Z}(G)$ [see Simon (1996), page 40]. This shows in particular that the number of classes is equal to the number $o(\hat{G})$ of irreducible representations. Observe however that, for an arbitrary finite group, there is no natural correspondence between the set of equivalence classes of G and the set $\hat{G}$ of irreducible representations.

Denote by $\chi^\alpha(c)$ the common value of the character χ^α on the class c. Another consequence of the orthonormality of characters is that if $C_G = (\chi^\alpha(c))$ is the square matrix of characters, then

$$\frac{1}{o(G)} \operatorname{Diag}(\sharp c) C_G^t = C_G^{-1}, \tag{6.2}$$

where $\sharp c$ is the number of elements of the class c.

To analyse the product $f * g$, recall that $f * g = 0$ if $f \in V_\alpha$, $g \in V_\beta$ and $\alpha \neq \beta$, that $D_{ij}^{(\alpha)} * D_{kl}^{(\alpha)} = 0$ if $j \neq k$ and that for all i, j, l in $1, 2, \dots, d_\alpha$,

$$D_{ij}^{(\alpha)} * D_{jl}^{(\alpha)} = \frac{o(G)}{d_\alpha} D_{il}^{(\alpha)}.$$

This implies in particular that $\chi^{(\alpha)} * D_{ij}^{(\alpha)} = \frac{o(G)}{d_\alpha} D_{ij}^{(\alpha)}$ and that $\chi^{(\alpha)} * \chi^{(\alpha)} = \frac{o(G)}{d_\alpha} \chi^{(\alpha)}$.

Now, suppose that f is in $\mathcal{Z}(G)$. We write $f = \sum_{\alpha \in \hat{G}} f_\alpha \chi^{(\alpha)}$, with

$$f_\alpha = \langle f, \chi^{(\alpha)} \rangle = \frac{1}{o(G)} \sum_c \overline{f(c)} \chi^{(\alpha)}(c) \sharp c, \tag{6.3}$$

where the sum is taken on the set of classes and where $f(c)$ is the common value of f on the class c.

Similarly, for g in $\mathcal{A}(G)$ we write $g = \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_\alpha} g_{\alpha,i,j} D_{ij}^{(\alpha)}$, with

$$g_{\alpha,i,j} = d_\alpha \langle g, D_{ij}^{(\alpha)} \rangle.$$

Thus we get

$$f * g = \sum_{\alpha \in \hat{G}} f_\alpha \frac{o(G)}{d_\alpha} \sum_{i,j=1}^{d_\alpha} g_{\alpha,i,j} D_{ij}^{(\alpha)}.$$

This equation clearly shows that the eigenspaces of the endomorphism $g \mapsto f * g = L(f)(g)$ of $\mathcal{A}(G)$ are the V_α and the eigenvalues are

$$f_\alpha \frac{o(G)}{d_\alpha} \tag{6.4}$$

with multiplicity d_α^2. Note that $\mathcal{Z}(G)$ is a stable subspace of $L(f)$ and that $L(f)$ can be considered as an endomorphism of $\mathcal{Z}(G)$ as well. Indeed, suppose that g is in $\mathcal{Z}(G)$, and write $g = \sum_{\alpha \in \hat{G}} g_\alpha \chi^{(\alpha)}$. Then

$$f * g = \sum_{\alpha \in \hat{G}} f_\alpha \frac{o(G)}{d_\alpha} g_\alpha \chi^{(\alpha)}.$$

Thus for $L(g)$ restricted to $\mathcal{Z}(G)$, the eigenvectors are the $\chi^{(\alpha)}$ and the eigenvalues obviously are again given by (6.4). Finally, we note that $L(f)$ is invertible if and only if for any α in $\hat{G}$ we have $f_\alpha \neq 0$. Under these conditions, we have $(L(f))^{-1} = L(f^{(-1)})$ with

$$f^{(-1)} = \sum_{\alpha\in\hat{G}} \frac{d_\alpha^2}{o(G)^2 f_\alpha} \chi^{(\alpha)}, \tag{6.5}$$

which satisfies $f * f^{(-1)} = \delta_e$.

We now specialize these notions to the group $\mathcal{S}_k$, of size $o(\mathcal{S}_k) = k!$.

For $k = 3$, there are three classes, $\{\pi_1\}$, $\{\pi_2, \pi_3\}$ and $\{\pi_4, \pi_5, \pi_6\}$, whose portraits are given by $\mathbf{i}(\pi_1) = (3, 0, 0)$, $\mathbf{i}(\pi_2) = (0, 0, 1)$ and $\mathbf{i}(\pi_4) = (1, 1, 0)$, abbreviated as (3), (0,0,1) and (1,1), respectively.

The functions from $\mathcal{S}_k$ to $\mathbf{C}$ given by $\pi \mapsto r_\pi(\sigma)(h_1, h_2, \dots, h_k)$ are examples of elements of $\mathcal{A}(\mathcal{S}_k)$. Note here an important fact: when $h_1 = \cdots = h_k = h$ the function $\pi \mapsto r_\pi(\sigma)(h, \dots, h)$ becomes a central function, that is an element of $\mathcal{Z}(\mathcal{S}_k)$. Other important examples of central functions are given by $\pi \mapsto m(\pi)$, p^m and $(-1)^k(-q)^m$. Since m is a class function, it can also be written $m(\pi) = m(\mathbf{i}) = i_1 + \cdots + i_k$ where $\mathbf{i}$ is the portrait of π. In terms of convolution, one can observe that Theorem 2 and Theorem 3 can be rewritten

$$\mathbb{E}(r(S)) = p^m * r(\sigma), \qquad r(\sigma^{-1}) = (-1)^k(-q)^m * \mathbb{E}\big(r(S^{-1})\big).$$

We consider now the set $\hat{G}$ of irreducible representations. It is certainly not possible to describe them all here [see Simon (1996), Chapter VI], but it is important to say that they are naturally indexed by the set M_k of sequences $\mathbf{m} = (m_1, \dots, m_k)$ of integers such that $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$ and such that $m_1 + \cdots + m_k = k$. For this reason, we choose to denote by $\chi^{\mathbf{m}}$ the character $\chi^{(\alpha)}$ of the corresponding irreducible representation. Since $\chi^{\mathbf{m}}$ is constant on the classes, we denote its value on π by $\chi^{\mathbf{m}}(\mathbf{i})$ when $\mathbf{i}$ is the portrait of π. Theoretical knowledge of the matrix (6.2) for the group $\mathcal{S}_k$, that is, of

$$C_k = \big(\chi^{\mathbf{m}}(\mathbf{i})\big)_{\mathbf{m}\in M_k, \mathbf{i}\in I_k}, \tag{6.6}$$

is obtained by the introduction of the *Schur polynomials* with k variables defined for $\mathbf{m}$ in M_k by

$$s_{\mathbf{m}}(z_1, \dots, z_k) = \frac{\det(z_i^{m_j+k-j})_{1\leq i,j\leq k}}{\det(z_i^{k-j})_{1\leq i,j\leq k}}. \tag{6.7}$$

The characters are now given by the *Frobenius formula* [see Simon (1996), Theorem VI 5.1]: For all $\mathbf{i}$ in I_k one has

$$\prod_{j=1}^{k} (z_1^j + \cdots + z_k^j)^{i_j} = \sum_{\mathbf{m}\in M_k} \chi^{\mathbf{m}}(\mathbf{i}) s_{\mathbf{m}}(z_1, \dots, z_k). \tag{6.8}$$

The expression for the dimension $d_{\mathbf{m}}$ of the corresponding irreducible representation has been given in (2.10).

Let us illustrate these notions for $k = 3$. We have

$$M_3 = \{(3,0,0),\ (2,1,0),\ (1,1,1)\} = \{(3),\ (2,1),\ (1,1,1)\}.$$

The Schur functions are

$$\begin{aligned} s_{(3)}(z_1,z_2,z_3) &= z_1^3 + z_2^3 + z_3^3 + (z_1 z_2^2 + 5 \text{ terms}) + z_1 z_2 z_3, \\ s_{(2,1)}(z_1,z_2,z_3) &= (z_1 z_2^2 + 5 \text{ terms}) + 2 z_1 z_2 z_3, \\ s_{(1,1,1)}(z_1,z_2,z_3) &= z_1 z_2 z_3. \end{aligned}$$

Ways to compute these functions for lower values of k are given in Fulton (1997). The table of characters of $\mathcal{S}_3$, together with the dimensions of the representations, is

(6.9)

$\mathbf{m}$	$(\mathbf{3})$	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(\mathbf{2},\mathbf{1})$
$d_{\mathbf{m}}$	1	1	2
$\chi^{\mathbf{m}}((3))$	1	1	2
$\chi^{\mathbf{m}}((1,1))$	1	-1	0
$\chi^{\mathbf{m}}((0,0,1))$	1	1	-1

We can now explain the form of $(M(q))^{-1}$, as given in (6.1) at the beginning of this section, by applying the above theory to $G = \mathcal{S}_3$ and to the element f of $\mathcal{Z}(\mathcal{S}_3)$ defined by $f((3)) = q^3$, $f((1,1)) = -q^2$ and $f((0,0,1)) = q$. Recall that (6.1) is the representative matrix of $L(f)^{-1}$. Following (6.3), we write $f = \sum_{\mathbf{m}\in M_3} f_{\mathbf{m}} \chi^{\mathbf{m}}$ with

$$f_{\mathbf{m}} = \frac{1}{6} \sum_{\mathbf{i}\in I_3} (-1)^3 (-q)^{m(\mathbf{i})} \chi^{\mathbf{m}}(\mathbf{i}) \sharp \mathbf{i}$$

and using the table of characters (6.9), we obtain

$$(f_{(3)}, f_{(1,1,1)}, f_{(2,1)}) = \left(\tfrac{1}{6}(q^3 - 3q^2 + 2q), \tfrac{1}{6}(q^3 + 3q^2 + 2q), \tfrac{1}{3}(q^3 - q)\right).$$

According to (6.4) the eigenvalues of $M(q)$ are therefore the ones given at the beginning of this section. We next want to compute $f^{(-1)}$. To do so, we use (6.5). The vector $(f^{(-1)}(\mathbf{i}))_{\mathbf{i}\in I_3}$ is obtained with the help of (6.9):

$$\begin{aligned} \begin{bmatrix} f^{(-1)}((3)) \\ f^{(-1)}((1,1)) \\ f^{(-1)}((0,0,1)) \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1/36 & 0 & 0 \\ 0 & 1/36 & 0 \\ 0 & 0 & 4/36 \end{bmatrix} \begin{bmatrix} 6/(q^3 - 3q^2 + 2q) \\ 6/(q^3 + 3q^2 + 2q) \\ 3/(q^3 - q) \end{bmatrix} \\ &= \begin{bmatrix} (q^2 - 2)/P(q) \\ q/P(q) \\ 2/P(q) \end{bmatrix}, \end{aligned}$$

with $P(q) = q(q^2 - 1)(q^2 - 4)$.

7. Proof of Theorem 4. For $r \le k$, denote by $M_{k,r}$ the set of sequences of integers $\mathbf{m} = (m_1, \dots, m_r)$ such that $m_1 \ge m_2 \ge \cdots \ge m_r \ge 0$ and such that $m_1 + \cdots + m_r = k$.

We now extend the definition of Schur polynomials $s_{\mathbf{m}}$, as given in (6.7) to the case of r variables and $\mathbf{m} \in M_{k,r}$ as follows:

$$s_{\mathbf{m}}(z_1, \dots, z_r) = s_{\mathbf{m}}(z_1, \dots, z_r, 0, \dots, 0) \tag{7.1}$$

[see Fulton (1997), Section 6.2]. Note that for $\mathbf{m} \in M_k \setminus M_{k,r}$ we have

$$s_{\mathbf{m}}(z_1, \dots, z_r, 0, \dots, 0) = 0. \tag{7.2}$$

We are now going to prove part (i) of Theorem 4 by showing the following, which is an identity between two polynomials with respect to the variable t. For all $\mathbf{i} \in I_k$, we have

$$t^{i_1+\cdots+i_k} = \sum_{\mathbf{m} \in M_k} \chi^{\mathbf{m}}(\mathbf{i}) \frac{d_{\mathbf{m}}}{k!} \prod_{j=1}^{k} \prod_{i=1}^{m_j} (t + i - j). \tag{7.3}$$

It is immediate to verify that, for $t = -q$, (7.3) is equivalent to (2.12). To prove (7.3), it suffices to fix $r \le k$ and let $z_1 = \cdots = z_r = 1$ and $z_{r+1} = \cdots = z_k = 0$ in the Frobenius formula (6.8). We are very grateful to William Fulton for giving us this idea. We thus obtain

$$\begin{aligned}
r^{i_1+\cdots+i_k} &\overset{(1)}{=} \sum_{\mathbf{m} \in M_k} \chi^{\mathbf{m}}(\mathbf{i}) s_{\mathbf{m}}(1, \dots, 1, 0, \dots, 0) \\
&\overset{(2)}{=} \sum_{\mathbf{m} \in M_{k,r}} \chi^{\mathbf{m}}(\mathbf{i}) s_{\mathbf{m}}(1, \dots, 1) \\
&\overset{(3)}{=} \sum_{\mathbf{m} \in M_{k,r}} \chi^{\mathbf{m}}(\mathbf{i}) \prod_{1 \le i < j \le r} \frac{m_i - m_j + j - i}{j - i} \\
&\overset{(4)}{=} \sum_{\mathbf{m} \in M_{k,r}} \chi^{\mathbf{m}}(\mathbf{i}) \prod_{j=1}^{k} \frac{d_{\mathbf{m}}}{k!} \prod_{i=1}^{m_j} (r + i - j) \\
&\overset{(5)}{=} \sum_{\mathbf{m} \in M_k} \chi^{\mathbf{m}}(\mathbf{i}) \frac{d_{\mathbf{m}}}{k!} \prod_{j=1}^{k} \prod_{i=1}^{m_j} (r + i - j).
\end{aligned}$$

In the above sequence of equalities, (1) comes from (6.8), (2) from (7.1) and (7.2), (3) from Fulton [(1997), Exercise 6, page 76] completed with formula (7), page 75 and Fulton's definition of Schur polynomials, page 3. Now, (4) comes from Stanley (1971), as indicated by Fulton (1997), formula (9), page 55. Finally, (5) comes from the easily verified fact that $\prod_{i=1}^{m_j} (r + i - j) = 0$ if $\mathbf{m}$ is in $M_k \setminus M_{k,r}$.

Thus (7.3) is true for $t = r = 1, 2, \dots, k$. Furthermore, it is trivially true for $t = 0$. Since both members of (7.3) are polynomials of degree $\le k$, (7.3) is proved, as well as (2.12).

The proof of part (ii) of Theorem 4 is now immediate. Since $q > k$, then for all $\mathbf{m}$ in M_k, we have $f_{\mathbf{m}} \neq 0$. Thus $f^{(-1)}$ as defined in (2.13) exists. From (6.5), and with the notation of Section 6 we have $f^{(-1)} * (-1)^k (-q)^m = \delta_e$. This proves (2.14) and Theorem 4.

COMMENT. For a given $\mathbf{m}$ in M_k, the polynomial in q equal to

$$\prod_{j=1}^{k} \prod_{i=1}^{m_j} (q + j - i)$$

is called the content polynomial. Many of its properties [but not (7.3)] appear in Macdonald (1995), in particular on pages 15 and 16.

8. Examples for $k = 4$ and 5. While in Section 5 we simply rediscovered results which had been previously obtained by brute force in the case $k = 3$, in this section we illustrate the usage of group theory to compute $\mathbb{E}(r_\pi(S^{-1})(h_1, \dots, h_k))$ for $k = 4$ and 5 [the calculations for $k = 6, 7, 8$ and 9 are detailed in Graczyk, Letac and Massam (2000)]. It involves square matrices of sizes 5 and 7, instead of 24 and 120, respectively. We will use tables of characters which can easily be found in the literature. [For $k = 4$ and 5, see, e.g., Simon (1996), pages 83 and 86, James and Kerber (1981), page 349, or Fulton and Harris (1991), pages 19 and 28. In this last reference, on page 28, lines W and W' should be exchanged.] Let f denote the element of $\mathcal{Z}(\mathcal{S}_k)$ defined by $f(\pi) = (-1)^k(-q)^{m(\pi)}$. For $k = 4$ the table of characters is

(8.1)

$\mathbf{m}$	$\mathbf{(4)}$	$\mathbf{(1,1,1,1)}$	$\mathbf{(3,1)}$	$\mathbf{(2,1,1)}$	$\mathbf{(2,2)}$
$\chi^{\mathbf{m}}((4))$	1	1	3	3	2
$\chi^{\mathbf{m}}((2,1))$	1	−1	1	−1	0
$\chi^{\mathbf{m}}((1,0,1))$	1	1	0	0	−1
$\chi^{\mathbf{m}}((0,0,0,1))$	1	−1	−1	1	0
$\chi^{\mathbf{m}}((0,2))$	1	1	−1	−1	2

Let C_4 be this $(5, 5)$ matrix. Recall that the $d_{\mathbf{m}}$ are nothing but $\chi^{\mathbf{m}}((k))$, that is, the first row of the above matrix. The size $\sharp\mathbf{i}$ of classes is given by (2.9). For $k = 4$ we have

$$\sharp(4) = 1, \quad \sharp(2,1) = 6, \quad \sharp(1,0,1) = 8, \quad \sharp(0,0,0,1) = 6, \quad \sharp(0,2) = 3$$

and the components of f in the basis of characters are given by

$$\begin{aligned}
&(f_{(4)}, f_{(1,1,1,1)}, f_{(3,1)}, f_{(2,1,1)}, f_{(2,2)}) \\
&\quad = (q^4, -q^3, q^2, -q, q^2)\tfrac{1}{24}\,\mathrm{Diag}(1,6,8,6,3)C_4 \\
&\quad = \big(\tfrac{1}{24}q(q-1)(q-2)(q-3), \tfrac{1}{24}q(q+1)(q+2)(q+3), \\
&\qquad \tfrac{1}{8}q(q^2-1)(q-2), \tfrac{1}{8}q(q^2-1)(q+2), \tfrac{1}{12}q^2(q^2-1)\big).
\end{aligned}$$

From this we can compute $f^{(-1)}$, which gives the solution to our inversion problem by $\mathbb{E}(r_\pi(S^{-1})) = (f^{(-1)} * r(\sigma^{-1}))_\pi$. It is enough to compute the column vector of the $(f^{-1}(\mathbf{i}))_{\mathbf{i}\in I_4}$. By formula (6.5) we get (we use t to indicate transposition)

$$\begin{aligned}
\big(f^{(-1)}(\mathbf{i})\big)_{\mathbf{i}\in I_4} &= C_4\frac{1}{24^2}\,\mathrm{Diag}(1,1,9,9,4) \\
&\qquad \times\bigg(\frac{24}{q(q-1)(q-2)(q-3)}, \frac{24}{q(q+1)(q+2)(q+3)}, \\
&\qquad \frac{8}{q(q^2-1)(q-2)}, \frac{8}{q(q^2-1)(q+2)}, \frac{12}{q^2(q^2-1)}\bigg)^t \\
&= \frac{(q^4-8q^2+6, q^3-4q, 2q^2-3, 5q, q^2+6)^t}{q^2(q^2-1)(q^2-4)(q^2-9)}.
\end{aligned}$$

For $k=5$ the character table is

(8.2)

$\mathbf{m}$	$\mathbf{(5)}$	$\mathbf{(1,1,1,1,1)}$	$\mathbf{(4,1)}$	$\mathbf{(2,1,1,1)}$	$\mathbf{(3,1,1)}$	$\mathbf{(3,2)}$	$\mathbf{(2,2,1)}$
$\chi^{\mathbf{m}}((5))$	1	1	4	4	6	5	5
$\chi^{\mathbf{m}}((3,1))$	1	−1	2	−2	0	1	−1
$\chi^{\mathbf{m}}((2,0,1))$	1	1	1	1	0	−1	−1
$\chi^{\mathbf{m}}((1,0,0,1))$	1	−1	0	0	0	−1	1
$\chi^{\mathbf{m}}((0,0,0,0,1))$	1	1	−1	−1	1	0	0
$\chi^{\mathbf{m}}((1,2))$	1	1	0	0	−2	1	1
$\chi^{\mathbf{m}}((0,1,1))$	1	−1	−1	1	0	1	−1

Let C_5 be this $(7,7)$ matrix of characters. For $k=5$, using (2.9) yields the following sizes for the classes:

$$\sharp(5)=1,\quad \sharp(3,1)=10,\quad \sharp(2,0,1)=20,\quad \sharp(1,0,0,1)=30,$$

$$\sharp(0,0,0,0,1)=24,\quad \sharp(1,2)=15,\quad \sharp(0,1,1)=20.$$

The components of f in the basis of characters are given by

$$
\begin{aligned}
&(f_{(\mathbf{m})})_{\mathbf{m}\in M_5}\\
&\quad = (q^5, -q^4, q^3, -q^2, q, q^3, -q^2)\tfrac{1}{120}\,\mathrm{Diag}(1, 10, 20, 30, 24, 15, 20)C_5\\
&\quad = \big(\tfrac{1}{120}q(q-1)(q-2)(q-3)(q-4), \tfrac{1}{120}q(q+1)(q+2)(q+3)(q+4),\\
&\qquad \tfrac{1}{30}q(q^2-1)(q-2)(q-3), \tfrac{1}{30}q(q^2-1)(q+2)(q+3),\\
&\qquad\qquad \tfrac{1}{20}q(q^2-1)(q^2-4), \tfrac{1}{24}q^2(q^2-1)(q-2), \tfrac{1}{24}q^2(q^2-1)(q+2)\big).
\end{aligned}
$$

The components of $f^{(-1)}$ are

$$
\begin{aligned}
&\big(f^{(-1)}(\mathbf{i})\big)_{\mathbf{i}\in I_5}\\
&\quad = C_5\frac{1}{120^2}\,\mathrm{Diag}(1, 1, 16, 16, 36, 25, 25)\\
&\qquad \times\bigg(\frac{120}{q(q-1)(q-2)(q-3)(q-4)}, \frac{120}{q(q+1)(q+2)(q+3)(q+4)},\\
&\qquad\quad \frac{30}{q(q^2-1)(q-2)(q-3)}, \frac{30}{q(q^2-1)(q+2)(q+3)},\\
&\qquad\qquad \frac{20}{q(q^2-1)(q^2-4)}, \frac{24}{q^2(q^2-1)(q-2)}, \frac{24}{q^2(q^2-1)(q+2)}\bigg)^t\\
&\quad = \frac{1}{q^2(q^2-1)(q^2-4)(q^2-9)(q^2-16)}\\
&\qquad \times(q^5-20q^3+78q, q^4-14q^2+24, 2q^3-18q,\\
&\qquad\qquad 5q^2-24, 14q, q^3-2q, 2q^2+24)^t.
\end{aligned}
$$

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