The hyperoctahedral group, symmetric group representations and the moments of the real Wishart distribution

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May 17, 2004

Abstract

In this paper, we compute all the moments of the real Wishart distribution. To do so, we use the Gelfand pair $(\mathcal{S}_{2k}, H)$, where H is the hyperoctahedral group, the representation theory of H and some techniques based on graphs.

1 Introduction

Let r be a positive integer and let M be the linear space of the (r, r) real matrices m, with unit element e. Let $V \subset M$ be the space of symmetric matrices s, let $\Omega \subset V$ be the cone of positive definite matrices and $\overline{\Omega} \subset V$ the cone of the non negative definite matrices. If

$$p \in \Lambda = \{\frac{1}{2}, \frac{2}{2}, \dots \frac{r-1}{2}, \} \cup (\frac{r-1}{2}, \infty),$$

and if $\sigma \in \Omega$, the Wishart distribution $\gamma_{p,\sigma}$ on $\overline{\Omega}$ is defined by its Laplace transform as follows: for $-\theta \in \Omega$, the Laplace transform of $\gamma_{p,\sigma}$ is

$$\int_{\overline{\Omega}} \exp \operatorname{tr}(\theta s) \gamma_{p,\sigma}(ds) = (\det(e - \theta\sigma))^{-p}.$$

For π an element of the group $\mathcal{S}_k$ of permutations of $\{1, \dots, k\}$, we write $m(\pi)$ for the number of cycles of π and $C(\pi)$ for the set of these cycles. For $m_1, \dots, m_k$ in M we denote

$$r_\pi(\sigma)(m_1, \dots, m_k) = \prod_{c \in C(\pi)} \operatorname{tr}(\prod_{j \in c} \sigma m_j).$$

We first observe that we have the following result.

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Proposition 1 *Let S follow the $\gamma_{p,\sigma}$ distribution and let $s_1, \ldots, s_k \in V$ be $r \times r$ real symmetric matrices. Then*

$$\mathbb{E}(\mathrm{tr}\,(Ss_1)\mathrm{tr}\,(Ss_2)\ldots\mathrm{tr}\,(Ss_k)) = \sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(s_1,\ldots,s_k). \tag{1.1}$$

Proof: For $p \in \Lambda$ and $\sigma \in \Omega$, we write

$$\gamma_{p,\sigma}(dx) = \exp(-\mathrm{tr}\,(x\sigma^{-1}))\,(\det\sigma)^{-p}\,\mu_p(dx)$$

where μ_p has Laplace transform

$$L(\theta) = (\det(-\theta))^{-p}, \quad \theta \in -\Omega.$$

For simplicity, we use the notation $\sigma(\theta) = (-\theta)^{-1}$. Standard arguments using induction show that the k-th differential of the map

$$\theta \;\to\; (\det\sigma(\theta))^p$$

is given by

$$(h_1,\ldots,h_k) \to (\det\sigma(\theta))^p \sum_{r\in\mathcal{S}_k} p^{m(r)} r_\pi(\sigma)(h_1,\ldots,h_k).$$

Since

$$\mathbb{E}(\mathrm{tr}\,(Sh_1)\ldots\mathrm{tr}\,(Sh_k)) = \frac{1}{L(\theta)}\,L^{(k)}(\theta)(h_1,\ldots,h_k)$$

the result follows immediately. More details can be found in [13] where Proposition 1 is proved in the real, complex and quaternionic cases. ∎

Formula (1.1) has been obtained in [6] where S follows the complex Wishart distribution and $s_1, \ldots, s_k$ can be not only in the space $V_{\mathbf{C}}$ of Hermitian $r \times r$ matrices but more generally in the space $M_{\mathbf{C}}$ of complex matrices. Let us point out that in the complex case, replacing $V_{\mathbf{C}}$ by the space $M_{\mathbf{C}}$ of complex matrices is a simple matter. It is sufficient to write

$$m_j = s_j + it_j$$

where $s_j, t_j \in V_{\mathbf{C}}$ and use multilinearity in order to obtain

$$\mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2)\ldots\mathrm{tr}\,(Sm_k)) = \sum_{\pi\in\mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(m_1,\ldots,m_k), \quad m_1,\ldots,m_k \in M_{\mathbf{C}}.$$

The reason why this is so is that the dimension r^2 of the **real** space $V_{\mathbf{C}}$ is equal to the **complex** dimension r^2 of $M_{\mathbf{C}}$. Therefore the complexification of $V_{\mathbf{C}}$ is simply $M_{\mathbf{C}}$. There is no such phenomenon in the real case.

When S is real Wishart, formula (1.1) is simply false if $m_1, \ldots, m_k \in M$ are not necessarily symmetric (see Corollary 11 for the exact formula which involves $(2k)!/(k!\,2^k)$ summands instead of $k!$ in (1.1)).

Also in the complex case, we have found, in [6], the surprising formula

$$\mathbb{E}(r_\pi(S)(m_1,\ldots,m_k)) = \sum_{\pi'\in\mathcal{S}_k} p^{m(\pi\circ\pi'^{,-1})} r_{\pi'}(\sigma)(m_1,\ldots,m_k) \;=\; (p^{m(.)} * r_.(\sigma))(\pi), \quad (1.2)$$

where the convolution is evaluated at π. This naturally leads to the question of whether in the real case we can compute

$$\mathbb{E}(r_\pi(S)(m_1,\ldots,m_k)) \;\text{ for }\; m_1,\ldots,m_k \in M$$

or at least

$$\mathbb{E}(r_\pi(S)(s_1,\ldots,s_k)) \;\text{ for }\; s_1,\ldots,s_k \in V.$$

There are two strong reasons why we should compute $\mathbb{E}(r_\pi(S)(s_1,\ldots,s_k))$. Clearly, the first is the search for an analog of (1.2). The second reason is more subtle.

Let $\mathcal{P}_k$ denote the space of functions P on V^{k+1}:

$$(x,s_1,\ldots,s_k) \;\mapsto\; P(x,s_1,\ldots,s_k)$$

such that
(i) for fixed x, the map $(s_1,\ldots,s_k) \mapsto P(x,s_1,\ldots,s_k)$ is a k-multilinear form on V^k
(ii) for fixed $(s_1,\ldots,s_k)$, the function $x \mapsto P(x,s_1,\ldots,s_k)$ is a homogeneous polynomial of degree k on V
(iii) P is invariant in the following sense: if $a \in M$ is invertible and a^* denotes the transposed of a then

$$P(axa^*,s_1,\ldots,s_k) = P(x,a^*s_1a,\ldots,a^*s_ka).$$

The application $r_\pi(x,s_1,\ldots,s_k) = r_\pi(x)(s_1,\ldots,s_k)$ is an element of $\mathcal{P}_k$ and in fact, we prove in the Appendix that the linear space $\mathcal{P}_k$ is generated by $(r_\pi)_{\pi\in\mathcal{S}_k}$.

Now, with $\gamma_{p,\sigma}$ denoting the real Wishart distribution, let us informally introduce the operator T_p on real functions on Ω as

$$T_p(f)(\sigma) = \int_\Omega f(x)\gamma_{p,\sigma}(dx) = \mathbb{E}(f(S)).$$

It is not difficult to verify that $T_p(f) \in \mathcal{P}_k$ if $f \in \mathcal{P}_k$. Therefore T_p is an endomorphism of $\mathcal{P}_k$ and $\mathbb{E}(r_\pi(S)(s_1,\ldots,s_k))$ is a linear combination of the $r_{\pi'}(\sigma)(s_1,\ldots,s_k))$, $\pi' \in \mathcal{S}_k$. To be more specific, if the dimension r is large enough with respect to k, then $(r_\pi)_{\pi\in\mathcal{S}_k}$ is a genuine basis of $\mathcal{P}_k$ and computing the coefficients of the $r_\pi(\sigma)(s_1,\ldots,s_k)$ in $\mathbb{E}(r_\pi(S))$ is equivalent to computing the matrix representative of T_p in the basis (r_π).

So obtaining $\mathbb{E}(r_\pi(S)(m_1,\ldots,m_k))$ is important but the question remains of why we should compute it for $m_1,\ldots,m_k$ in M rather than in V. The answer is that direct computation with $m_i \in V, i = 1,\ldots,k$ seems to be impossible.

We proceed as follows. We first express

$$\mathbb{E}(r_\pi(S)(m_1,\ldots,m_k))$$

as a linear combination of $(2k)!/(2^k k!)$ functions taken among the $2^k k!$ functions

$$r_\pi(\sigma)(m_1^{(\epsilon_1)}, \ldots, m_k^{(\epsilon_k)})$$

where $(\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k$, with the convention $m^{(1)} = m$ and $m^{(-1)} = m^t$, the transpose of m. Then we give the expression of this formula when $m_j = s_j$, a symmetric matrix. As mentioned before, a direct computation of $\mathbb{E}(r_\pi(S)(s_1, \ldots, s_k))$ by the methods used to compute $\mathbb{E}(r_\pi(S)(m_1, \ldots, m_k))$ seems difficult. Moreover, the computation of $\mathbb{E}(r_\pi(S)(m_1, \ldots, m_k))$ for $m_j \in M$ rather than $m_j \in V$ gives insight into the structure of the problem. We will see that this structure is actually not governed by the symmetric group $\mathcal{S}_k$ but rather by the Gelfand pair $(\mathcal{S}_{2k}, H)$ where H is the subgroup of the permutations $\pi \in \mathcal{S}_{2k}$ such that for all $j = 1, \ldots, k$

$$|\pi(j) - \pi(j + k)| = k.$$

The subgroup H is sometimes denoted in the literature by B_k and is called the hyperoctahedral group. It is isomorphic to the isometry group of the hyperoctahedron in the Euclidean space $\mathbf{R}^k$, namely the polyhedron with vertices $\pm e_i$ where $e_1, \ldots, e_k$ is the canonical basis of $\mathbf{R}^k$. Note that H is not a normal subgroup of $\mathcal{S}_{2k}$.

The paper is organized as follows. Section 2 contains combinatorial properties of the pair $(\mathcal{S}_{2k}, H)$ that we shall need later. In Section 3 we compute the polynomial coefficients $A(p)$ and $B(p)$ in the formulas

$$\mathbb{E}(r_\pi(S)(m_1^{\epsilon_1}, \ldots, m_k^{\epsilon_k}))) = \sum_{(\pi', \epsilon')} A_{(\pi,\epsilon),(\pi',\epsilon')}(p) r_{\pi'}(\sigma)(m_1^{\epsilon'_1}, \ldots, m_k^{\epsilon'_k}) \tag{1.3}$$

$$\mathbb{E}(r_\pi(S)(s_1, \ldots, s_k)) = \sum_{\pi' \in \mathcal{S}_k} B_{\pi,\pi'}(p) r_{\pi'}(\sigma)(s_1, \ldots, s_k). \tag{1.4}$$

in terms of a certain operation $(a, b) \to a \otimes b = c$ on the classes $a, b \in \mathcal{S}_{2k}/H$ and the inverse operation $c \oslash a$. In Section 4, with the help of graphs, we design a method for computing $a \otimes b = c$ and for solving $a \otimes x = c$ when a and c are given. In Section 5, we give a group-theoretical interpretation of the matrix A by using the representation theory of the symmetric group $\mathcal{S}_{2k}$. Whereas in the complex case, we only use the convolution on $\mathcal{S}_k$ in order to obtain (1.2), in the real case we need the convolution on $\mathcal{S}_{2k}/H$ in the sense of a Gelfand pair. We are also led to use the H-spherical representations of $\mathcal{S}_{2k}$ as associated to some convenient Young tableaux, a finer tool than the character table of $\mathcal{S}_{2k}$. In the Appendix we give proofs for the properties of the space $\mathcal{P}_k$ that we mentioned above and we make some further comments.

Let us mention that in [14] a formula for the moments of arbitrary polynomials in the entries of S was obtained as an application of MacMahon's Master Theorem. However our formulas for $\mathbb{E}(r_\pi(S)(m_1, \ldots, m_k))$ cannot be deduced from that one.

2 The homogeneous space $\mathcal{S}_{2k}/H$

In this section, we do the following:

• we define the hyperoctahedral subgroup H of $\mathcal{S}_{2k}$ and the homogeneous space $\mathcal{S}_{2k}/H$

• we show we need H because the map

$$(\pi, \epsilon) \mapsto r_\pi(\sigma)(m_1^{\epsilon_1}, \ldots, m_k^{\epsilon_k}) \tag{2.5}$$

is not one to one

• we study the properties of $\mathcal{S}_{2k}/H$ used in the computation of the moments $\mathbb{E}(r_\pi(S)(m_1^{\epsilon_1}, \ldots, m_k^{\epsilon_k}))$.

Let us first introduce some notation. For $j = 1, \ldots, k$, we write j^+ for $j + k$. For $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k = \mathbf{Z}_2^k$, τ_ϵ denotes the element of $\mathcal{S}_{2k}$ defined as follows: for all $j = 1, \ldots, k$,

$$\begin{array}{rlll} \text{if } \epsilon_j = 1, & \tau_\epsilon(j) = j & \text{and} & \tau_\epsilon(j^+) = j^+ \\ \text{if } \epsilon_j = -1, & \tau_\epsilon(j) = j^+ & \text{and} & \tau_\epsilon(j^+) = j \quad . \end{array}$$

If we write the elements $\{1, \ldots, 2k\}$ as

$$\begin{pmatrix} 1 & 2 & \ldots & j & \ldots & k \\ 1^+ & 2^+ & \ldots & j^+ & \ldots & k^+ \end{pmatrix}$$

then τ_ϵ acts by transposing the elements of some columns, e.g. for $k = 5$

$$\tau_{(1,-1,-1,1,-1)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \end{pmatrix} = \begin{pmatrix} 1 & 2^+ & 3^+ & 4 & 5^+ \\ 1^+ & 2 & 3 & 4^+ & 5 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 8 & 4 & 10 \\ 6 & 2 & 3 & 9 & 5 \end{pmatrix}.$$

The map $\epsilon \mapsto \tau_\epsilon$ is an injective homomorphism of the commutative group $\mathbf{Z}_2^k$ (with the product $\epsilon\epsilon' = (\epsilon_1\epsilon_1', \ldots, \epsilon_k\epsilon_k')$) into $\mathcal{S}_{2k}$. Its image is the subgroup $\mathcal{S}_{\{1,1^+\}} \times \cdots \times \mathcal{S}_{\{k,k^+\}}$ of $\mathcal{S}_{2k}$. If $\pi \in \mathcal{S}_k$, denote by t_π and s_π the elements of $\mathcal{S}_{2k}$ defined as follows: for all $j = 1, \ldots, k$,

$$t_\pi(j) = j, \quad t_\pi(j^+) = \pi(j)^+,$$

$$s_\pi(j) = \pi(j), \quad s_\pi(j^+) = \pi(j)^+.$$

We see that the permutations t_π are therefore nothing but the elements of the subgroup $\mathcal{S}_{\{1^+,\ldots,k^+\}} = \mathcal{S}_{\{1+k,\ldots,2k\}}$ of $\mathcal{S}_{2k}$.

The permutations s_π act on $\begin{pmatrix} 1 & 2 & \ldots & k \\ 1^+ & 2^+ & \ldots & k^+ \end{pmatrix}$ by permuting the columns, e.g.

$$s_{(123)(45)} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 & 5 & 4 \\ 2^+ & 3^+ & 1^+ & 5^+ & 4^+ \end{pmatrix}.$$

Both $\pi \mapsto t_\pi$ and $\pi \mapsto s_\pi$ are injective homomorphisms of $\mathcal{S}_k$ into $\mathcal{S}_{2k}$. Note that $t(\mathcal{S}_k) \cup \tau(\mathbf{Z}_2^k)$ generates $\mathcal{S}_{2k}$.

Definition 2 *The hyperoctahedral group H is the subgroup of $\mathcal{S}_{2k}$ generated by all the permutations τ_ϵ, $\epsilon \in \mathbf{Z}_2^k$ and s_π, $\pi \in \mathcal{S}_k$.*

The hyperoctahedral group H is called a *wreath product* of $\mathcal{S}_2$ and $\mathcal{S}_k$ (the image of τ is to be identified with $\mathcal{S}_2^{\otimes k}$ and the image of s with $\mathcal{S}_k$). Note that $(\mathcal{S}_{2k}, H)$ is a Gelfand pair (cf.[7]).

We gather the properties of H in the following proposition. Its proof is easy and we omit it.

Proposition 3 *For (ϵ, π) in $\mathbf{Z}_2^k \times \mathcal{S}_k$, define*

$$b_\pi(\epsilon) = (\epsilon_{\pi^{-1}(1)}, \ldots, \epsilon_{\pi^{-1}(k)}).$$

Then

(i) $\pi \mapsto b_\pi(\epsilon)$ is a homomorphism of $\mathcal{S}_k$ into the group of automorphisms of $\mathbf{Z}_2^k$,

(ii) the following commutation rule holds : $\tau_{b_\pi(\epsilon)} s_\pi = s_\pi \tau_\epsilon$,

(iii) the subgroup H is equal to

$$H = \{s_\pi \tau_\epsilon,\ (\pi, \epsilon) \in \mathcal{S}_k \times \mathbf{Z}_2^k\}\ . \tag{2.6}$$

Moreover, the group H is isomorphic to the semi-direct product of the groups $\mathcal{S}_k$ and $\mathbf{Z}_2^k$ induced by the action $\pi \mapsto b_\pi(\epsilon)$ of $\mathcal{S}_k$ on $\mathbf{Z}_2^k$.

The subgroup H has $2^k k!$ elements. It is not normal. We denote by $\mathcal{S}_{2k}/H = \{C_1 \ldots, C_N\}$, with $N = \frac{(2k)!}{2^k k!} = 1.3.5 \ldots (2k-1)$, the family of its right cosets, i.e. the family of the subsets of $\mathcal{S}_{2k}$ of the form gH and we write

$$g \sim g'$$

if there exists $h \in H$ such that $g' = gh$, i.e. if g and g' are in the same coset.

Proposition 4 *Let (ϵ, π) and (ϵ', π') be in $\mathbf{Z}_2^k \times \mathcal{S}_k$. Then $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$ if and only if the following conditions hold*

1. *The fonction $j \mapsto \epsilon_j \epsilon'_j$ is constant on the cycles of π.*
2. *If $\epsilon_j \epsilon'_j = 1$ on a cycle c of π, then π and π' coincide on c.*
3. *If $\epsilon_j \epsilon'_j = -1$ on a cycle c of π, then π^{-1} and π' coincide on c.*

Proof: By Proposition 3 (iii) $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$ if and only if there exist π'' and ϵ'' such that

$$\tau_\epsilon t_\pi = \tau_{\epsilon'} t_{\pi'} \tau_{\epsilon''} s_{\pi''},$$

or equivalently

$$t_\pi = \tau_{\epsilon\epsilon'} t_{\pi'} \tau_{\epsilon''} s_{\pi''}. \tag{2.7}$$

Thus, without loss of generality, we may assume that ϵ is the identity of $\mathbf{Z}_2^k$. Let us fix j in $\{1, 2, \ldots, k\}$. We apply (2.7), with $\epsilon = id$, to j and j^+ and we discuss the different cases according to the value of $\epsilon''_{\pi''(j)}$.

First case: if $\epsilon''_{\pi''(j)} = 1$, we first apply both sides of (2.7) to j. We then obtain $\tau_{\epsilon'}(\pi'')(j) = j$ which implies that $\epsilon'_{\pi''(j)} = 1$ and $\pi''(j) = j$.

From the equality $\pi''(j) = j$ and the application of (2.7) to j^+ we get $\tau_{\epsilon'}(\pi'(j)^+) = \pi(j)^+$, hence $\epsilon'_{\pi'(j)} = 1$ and $\pi(j) = \pi'(j)$. Thus, in this case, $\epsilon'_j = \epsilon'_{\pi(j)} = 1$ and $\pi(j) = \pi'(j)$.

Second case: if $\epsilon''_{\pi''(j)} = -1$, from the action on j^+, we have $\pi(j)^+ = \tau_{\epsilon'}(\pi''(j))$, so that $\epsilon'_{\pi''(j)} = -1$ and $\pi(j) = \pi''(j)$. From the action on j, we obtain $j = \tau_{\epsilon'}(\pi'\pi(j)^+)$ and we deduce that $\epsilon'_{\pi'\pi(j)} = -1$ and $\pi'\pi(j) = j$. Thus, in this case, $\epsilon'_j = \epsilon'_{\pi(j)} = -1$ and $\pi(j) = \pi'^{-1}(j)$.

It is therefore clear that the function $j \mapsto \epsilon'_j$ is constant on the cycles of π. Conditions 2 and 3 now also follow immediately.

The converse part of the proposition is easy to verify. ∎

We are now going to explain how the space $\mathcal{S}_{2k}/H$ is related to (2.5). For fixed k, let D be the set of sequences $B = (c_1, \ldots, c_{2k})$ of elements of $\{1, \ldots, r\}$. For $g \in \mathcal{S}_{2k}$, we denote l_g the permutation of D defined by $l_g(B) = (c_{g(1)}, \ldots, c_{g(2k)})$. We will write

$$l_g(B) = \begin{pmatrix} c_{g(1)} & \ldots & c_{g(k)} \\ c_{g(k+1)} & \ldots & c_{g(2k)} \end{pmatrix}.$$

The map $g \mapsto l_g$ is an injective antihomomorphism of $\mathcal{S}_{2k}$, that is $l_g \circ l_{g'} = l_{g'g}$.

Elements of D will also frequently be denoted by $B = \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix}$. For such a B and for a matrix $m = (m_{ij})_{1 \le i,j \le r}$, we denote

$$m(B) = m_{a_1 b_1} \ldots m_{a_k b_k}.$$

Thus $m(l_g(B))$ makes sense. In order to compute $m(l_g(B))$ for $g = \tau_\epsilon t_\pi$, we introduce the basis $\{e(a,b)\}_{1 \le a,b \le r}$ of M where, if $1 \le a, b \le r$, $e(a,b)$ is the matrix such that $e(a,b)_{b,a} = 1$ and $e(a,b)_{t,s} = 0$ if $(t,s) \ne (b,a)$. This basis has the property that $\mathrm{tr}\,(me(a,b)) = m_{ab}$ for $m \in M$, and that for $B = \begin{pmatrix} a_1 & \ldots & a_k \\ b_1 & \ldots & b_k \end{pmatrix}$

$$m(B) = \mathrm{tr}\,(me(a_1,b_1)) \ldots \mathrm{tr}\,(me(a_k,b_k)). \tag{2.8}$$

If $\pi \in \mathcal{S}_k$ is a circular permutation, say $\pi(j) = j+1$ for $j = 1, \ldots, k-1$ and $\pi(k) = 1$, then([6], (4.28))

$$m(l_{t_\pi}(B)) = \mathrm{tr}\,(me(a_1,b_1) \ldots me(a_k,b_k)).$$

More generally, for $\pi \in \mathcal{S}_k$ we have([6], (4.29))

$$m(l_{t_\pi}(B)) = r_\pi(m)(e(a_1,b_1), \ldots, e(a_k,b_k)) \tag{2.9}$$

Recall that we have denoted by $m^{(-1)}$ the transposed matrix of m. Thus $e(a,b)^{(-1)} = e(b,a)$ and we can write

$$m(l_{\tau_\epsilon t_\pi}(B)) = m(l_{t_\pi} l_{\tau_\epsilon}(B)) = r_\pi(m)(e(a_1,b_1)^{\epsilon_1}, \ldots, e(a_k,b_k)^{\epsilon_k}). \tag{2.10}$$

Proposition 5 *If $g \sim g'$ then for all symmetric $s \in V$ and for all $h \in H$,*

$$s(l_h(B)) = s(B). \tag{2.11}$$

Proof: Let us observe that for $\pi \in \mathcal{S}_k$

$$m(l_{s_\pi}(B)) = m(B)$$

and for $\epsilon \in \mathbf{Z}_2^k$ and s symmetric

$$s(l_{\tau_\epsilon}(B)) = s(B).$$

These imply that for $s \in V$ and $h \in H$

$$s(l_h(B)) = s(B). \tag{2.12}$$

Remark now that if $g \sim g'$, i.e. if there exists h in H such that $g = g'h$, then for B in D and for s in V we have

$$s(l_g(B)) = s(l_{g'}(B)). \tag{2.13}$$

To see this, introduce $B_1 = l_{g'}(B)$ so that $B = l_{g'^{-1}}(B_1)$. Then

$$l_g(B) = l_g(l_{g'^{-1}}(B_1)) = l_{g'^{-1}g}(B_1) = l_h(B_1)$$

and it follows from (2.11) that $s(l_h(B_1)) = s(B_1)$. ∎

Corollary 6 *The equality*

$$r_\pi(\sigma)(m_1^{\epsilon_1}, \ldots, m_k^{\epsilon_k}) = r_{\pi'}(\sigma)(m_1^{\epsilon'_1}, \ldots, m_k^{\epsilon'_k}) \tag{2.14}$$

holds for all symmetric matrices $\sigma \in V$ and all real matrices $m_1, \ldots, m_k$ if and only if

$$\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}.$$

Proof: Sufficiency of (2.14) follows from Proposition 5 and from (2.10), by multilinearity of $r_\pi(\sigma)$. To prove that (2.14) is necessary, let us observe that (2.11) holds for all $s \in V$ if and only if $h \in H$. Set $\tau_\epsilon t_\pi \approx \tau_{\epsilon'} t_{\pi'}$ when (2.14) holds. It follows that this equivalence relation on $\{\tau_\epsilon t_\pi \mid (\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k\}$ is the same as $\sim$: in fact, the equivalence classes of $((1, 1, \ldots, 1), \mathrm{id})$ are equal to $H \cap \{\tau_\epsilon t_\pi \mid (\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k\}$ for both relations. ∎

From now on, if C is in $\mathcal{S}_{2k}/H$, B is in D and s is in V, we define $s(C(B))$ to be $s(l_g(B))$ where g is an arbitrary element of C.

We now return to the properties of the space $\mathcal{S}_{2k}/H$ which will be useful in the computation of the moments of the Wishart distribution. Our aim is to show that for each coset C_i of $\mathcal{S}_{2k}/H$ we may choose a representative of the form $\tau_\epsilon t_\pi$.

The following theorem analyses the position of the set of permutations of this form with respect to $\mathcal{S}_{2k}/H$. For certain pairs (ϵ, π) that we will call *particular*, the corresponding permutations play a special role. They will be used to parametrize $\mathcal{S}_{2k}/H$ in a one-to-one way.

Definition 7 *We will say that a pair $T = (\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k$ is* **particular** *if, for any cycle c of π, we have $\epsilon_j = 1$ when j is the smallest element of c. We will call the corresponding permutation $T' = \tau_\epsilon t_\pi$* **particular**.

Theorem 8 *The class of $\mathcal{S}_{2k}/H$ containing $\tau_\epsilon t_\pi$ has exactly $2^{m(\pi)}$ elements of the type $\tau_{\epsilon'} t_{\pi'}$. For a given $\pi \in \mathcal{S}_k$, there are $2^{k-m(\pi)}$ distinct classes of $\mathcal{S}_{2k}/H$ containing an element of the form $\tau_\epsilon t_\pi$.*

Every class of $\mathcal{S}_{2k}/H$ contains exactly one particular permutation of the form $\tau_\epsilon t_\pi$.

Proof: By Proposition 4, for a given (ϵ, π), the number of ϵ' such that $\epsilon\epsilon'$ is constant on the cycles of π is exactly $2^{m(\pi)}$. It follows immediately that the first two statements hold.

To prove the last one, we recall that the *portrait* of a permutation $\pi \in \mathcal{S}_k$ is the sequence $\mathbf{i} = (i_1, \ldots, i_k)$ of non negative integers such that i_j is the number of cycles of π of length j. Therefore $m(\pi) = i_1 + \cdots + i_k$ and $1i_1 + 2i_2 + \cdots + ki_k = k$. The number of $\pi \in \mathcal{S}_k$ with a given portrait $\mathbf{i}$ is

$$\sharp\mathbf{i} = \frac{k!}{\prod_{j=1}^k i_j! j^{i_j}}.$$

We now prove the following equality

$$\sum_{\pi \in \mathcal{S}_k} 2^{k-m(\pi)} = N = \frac{(2k)!}{2^k k!},$$

which is equivalent to

$$\sum_{i_1,\ldots,i_k;\ 1i_1+2i_2+\cdots+ki_k=k} \frac{k!}{\prod_{j=1}^k i_j! j^{i_j}} 2^{k-i_1-\cdots-i_k} = \frac{(2k)!}{2^k k!}. \tag{2.15}$$

To prove this equality, let us denote by a_k its left hand side and by b_k its right hand side, and let us show that $\sum_{k=0}^\infty a_k z^k/k! = \sum_{k=0}^\infty b_k z^k/k!$ for $|z|$ small enough. It is easy to verify that $\sum_{k=0}^\infty b_k z^k/k! = (1-2z)^{-1/2}$. For the a_k's, we write

$$\begin{aligned}
\sum_{k=0}^\infty \frac{a_k z^k}{k!} &= \sum_{i_1,\ldots,i_k,\ldots} \prod_{j=1}^\infty \frac{2^{i_j(j-1)} z^{i_j}}{i_j! j^{i_j}} \\
&= \sum_{i_1,\ldots,i_k,\ldots} \exp\left(\frac{z}{1} + \frac{2^{2-1}z^2}{2} + \cdots + \frac{2^{j-1}z^j}{j} + \cdots\right) \\
&= (1-2z)^{-1/2},
\end{aligned}$$

where the infinite sum $\sum_{i_1,\ldots,i_k,\ldots}$ is taken on the countable set of sequences $i_1, \ldots, i_k, \ldots$ such that $i_k = 0$ for k large enough.

It follows from Definition 7 that given π in $\mathcal{S}_k$, the number of particular pairs (ϵ, π) is $2^{k-m(\pi)}$. We are going to show that

(i) if (ϵ, π) and (ϵ', π) are two distinct particular pairs, then $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$ is impossible.

(ii) for all $(\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k$, there exists a particular pair (ϵ', π') such that $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$.

Indeed, suppose that (i) is false. Then $\epsilon\epsilon'$ is a constant function on the cycles of π. Since the pairs are particular, there exists on each cycle a j such that $\epsilon_j = \epsilon'_j = 1$, therefore $\epsilon = \epsilon'$. This implies $\pi = \pi'$ and the pairs are equal, which is a contradiction.

For (ii), given (ϵ, π), we take ϵ' such that $\epsilon\epsilon'$ is a constant function on the cycles of π and such that $\epsilon'_j = 1$ when j is the smallest element of a cycle. We define π' to be equal to π on the cycles on which $\epsilon\epsilon' = 1$, and equal to π^{-1} on the others. Clearly the pair (ϵ', π') is particular and $\tau_\epsilon t_\pi \sim \tau_{\epsilon'} t_{\pi'}$.

This shows that the set of particular pairs is in a 1-1 correspondence with the elements of $\mathcal{S}_{2k}/H$ containing some $\tau_\epsilon t_\pi$. Identity (2.15) shows that the number of particular pairs is N and the proof of Theorem 8 is complete. ∎

We denote

$$c_i = \{(\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k \mid \tau_\epsilon t_\pi \in C_i\} \neq \emptyset, \tag{2.16}$$

and write

$$T_i \in c_i$$

for the only particular pair T_i which, according to Theorem 8, belongs to the set c_i.

Examples: For $k = 2$ we have $N = 3$ classes. Let e the identity and $d = (12)$ be the two elements of $\mathcal{S}_2$.

The sets c_i defined by (2.16) corresponding to the three classes C_1, C_2, C_3, are:

$$\begin{aligned} c_1 &= \{(\epsilon, e) \mid \epsilon \in \mathbf{Z}_2^2\} \\ c_2 &= \{((1,1), d),\ ((-1,-1), d)\} \\ c_3 &= \{((1,-1), d),\ ((-1,1), d)\} \end{aligned}$$

The corresponding particular pairs are $T_1 = ((1,1), e),\ T_2 = ((1,1), d), T_3 = ((1,-1), d)$.

For $k = 3$ we have $N = 15$ classes.

The particular pairs are:

$$\begin{aligned} T_1 &= ((1,1,1), e), \\ T_2 &= ((1,1,1), (1)(23)), & T_3 &= ((1,1,-1), (1)(23)), \\ T_4 &= ((1,1,1), (2)(13)), & T_5 &= ((1,1,-1), (2)(13)), \\ T_6 &= ((1,1,1), (3)(12)), & T_7 &= ((1,-1,1), (3)(12)), \\ T_8 &= ((1,1,1), (123)), & T_9 &= ((1,1,-1), (123)), \\ T_{10} &= ((1,-1,1), (123)), & T_{11} &= ((1,-1,-1), (123)), \\ T_{12} &= ((1,1,1), (132)), & T_{13} &= ((1,1,-1), (132)), \\ T_{14} &= ((1,-1,1), (132)), & T_{15} &= ((1,-1,-1), (132)). \end{aligned}$$

The set $c_1 = \{(\epsilon, e);\ \epsilon \in \mathbf{Z}_2^3\}$ has $2^3 = 8$ elements.

Each of the 6 sets $c_2, \ldots, c_7$ has $2^2 = 4$ elements. For example

$$c_2 = \{((\pm 1, \pm(1,1)), (1)(23))\}, \quad c_3 = \{(((\pm 1, \pm(1,-1)), (1)(23))\}\ .$$

The sets c_4, c_5, c_6, c_7 are constructed in an analogous manner.

Each of the 8 sets $c_8, \ldots, c_{15}$ contains $2^1 = 2$ elements and they are all of the form

$$\{(\epsilon, (123)),\ (-\epsilon, (132))\},\ \epsilon \in \mathbf{Z}_2^3$$

(note that $(123)^{-1} = (132)$). The $2^k k! = 48$ elements $t_\pi \tau_\epsilon$ are distributed among the sets $c_1, \ldots, c_{15}$, as described above .

3 Basic formulas

Proposition 9 *Let B be in D and let S follow the $\gamma_{p,\sigma}$ distribution on $\bar{\Omega}$. Then*

$$I\!E(S(B)) = \frac{1}{2^k} \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} \sum_{\epsilon \in \mathbf{Z}_2^k} \sigma(l_{\tau_\epsilon t_\pi}(B)).$$

Proof: Consider the symmetric matrix $h(a,b) = \frac{1}{2}(e(a,b)+e(b,a))$. For $B = (a_1, b_1, \ldots, a_k, b_k)$, we can then write

$$\begin{aligned}
\mathbb{E}(S(B)) &\overset{(1)}{=} \mathbb{E}(S_{a_1 b_1} \ldots S_{a_k b_k}) \\
&\overset{(2)}{=} \mathbb{E}(\mathrm{tr}\,(Se(a_1,b_1)) \ldots \mathrm{tr}\,(Se(a_k,b_k))) \\
&\overset{(3)}{=} \mathbb{E}(\mathrm{tr}\,(Sh(a_1,b_1)) \ldots \mathrm{tr}\,(Sh(a_k,b_k))) \\
&\overset{(4)}{=} \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} r_\pi(\sigma)(h(a_1,b_1), \ldots, h(a_k,b_k)) \\
&\overset{(5)}{=} \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} \frac{1}{2^k} \sum_{\epsilon \in \mathbf{Z}_2^k} r_\pi(\sigma)(e(a_1,b_1)^{\epsilon_1}, \ldots, e(a_k,b_k)^{\epsilon_k}) \\
&\overset{(6)}{=} \frac{1}{2^k} \sum_{\pi \in \mathcal{S}_k} p^{m(\pi)} \sum_{\epsilon \in \mathbf{Z}_2^k} \sigma(l_{\tau_\epsilon t_\pi}(B)).
\end{aligned}$$

In this list of equalities, (1) comes from the definition of $S(B)$, (2) from (2.8), (3) from the symmetry of S, (4) from Proposition 1, (5) from linearity, and (6) from (2.10). ∎

The formula in Proposition 9 can be improved by observing that the numbers $\sigma(l_{\tau_\epsilon t_\pi}(B))$ are not all distinct: there are $2^k k!$ pairs (ϵ, π) and only $N < 2^k k!$ possible values $\sigma(C_i(B))$ of these numbers. By a slight abuse of notation we will write

$$\sigma(T(B)) = \sigma(l_{\tau_\epsilon t_\pi}(B)) = \sigma(l_{T'}(B)) = \sigma(C(B))$$

for a particular pair $T = (\epsilon, \pi)$ such that $T' = \tau_\epsilon t_\pi \in C \in \mathcal{S}_{2k}/H$. We will also write

$$m(j) = m(\pi_j)$$

for a particular pair $T_j = (\epsilon_j, \pi_j)$.

Theorem 10 *Let B be in D and S be $\gamma_{p,\sigma}$ distributed on $\bar{\Omega}$. Then*

$$I\!E(S(B)) = \sum_{j=1}^{N} \frac{p^{m(j)}}{2^{k-m(j)}} \sigma(T_j(B)). \tag{3.17}$$

Proof: Proposition 9 implies that

$$\mathbb{E}(S(B)) = \sum_{j=1}^{N} a_j \sigma(T_j(B)).$$

where, using the notation c_i defined in (2.16)

$$a_j = \sum_{(\epsilon,\pi)\in c_j} \frac{p^{m(\pi)}}{2^k}.$$

However, Proposition 4 shows that all π's such that there exists ϵ satisfying $(\epsilon, \pi) \in c_j$ have the same number of cycles $m(j)$. Furthermore, Theorem 8 implies that c_j has $2^{m(j)}$ elements. Thus finally

$$a_j = \frac{p^{m(j)}}{2^{k-m(j)}} \tag{3.18}$$

and therefore formula (3.17) is proved. ∎

We observe that Theorem 10 gives us a quick way to compute any moments of the form

$$\mathbb{E}(S_{a_1b_1} S_{a_2b_2} \dots S_{a_kb_k}).$$

We have $S_{ab} = \mathrm{tr}\,(Se_{ab})$ and (2.10) together with Theorem 10 imply the following.

Corollary 11 *For any matrices $m_1, \dots, m_k \in M$ we have*

$$I\!E(\mathrm{tr}\,(Sm_1) \dots \mathrm{tr}\,(Sm_k)) = \sum_{j=1}^{N} \frac{p^{m(j)}}{2^{k-m(j)}} r_{\pi_j}(\sigma)(m_1^{\epsilon_1(j)}, \dots m_k^{\epsilon_k(j)})$$

where $(\epsilon(j), \pi_j) = T_j$, $j = 1, \dots, N$ are the particular pairs.

Example. A list of particular pairs in the case $k = 3$ can be found in the examples of Section 2. An application of Corollary 11 gives

$$\begin{aligned}
\mathbb{E}(\mathrm{tr}\,(Sm_1)(\mathrm{tr}\,(Sm_2)\mathrm{tr}\,(Sm_3)) &= p^3 \mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_3) \\
+\frac{p^2}{2}[\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3) &+ \mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3^t) + \mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3) \\
+\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3^t) &+ \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2) + \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2^t)] \\
+\frac{p}{4}[\mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3) &+ \mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3) + \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3^t) \\
+\mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2) &+ \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2^t)].
\end{aligned}$$

Definition 12 *Let $T_1, \dots, T_N$ be an order of the particular pairs and $T'_1, \dots, T'_N$ the order of the corresponding particular permutations $T'_j = \tau_{\epsilon(j)} t_{\pi_j}$. For $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, N\}$ denote by*

$$n = i \otimes j$$

the unique $n \in \{1, \dots, N\}$ such that

$$T'_i T'_j \sim T'_n$$

i.e. $T_i' T_j' \in C_n$. *Since for fixed* $i \in \{1, \ldots, N\}$ *the map* $j \mapsto i \otimes j$ *from* $\{1, \ldots, N\}$ *to itself is injective, we denote by*

$$n \oslash i = j$$

the unique j *such that*

$$n = i \otimes j.$$

We now have the following formulas for the moments of the Wishart distribution:

Theorem 13 *Let* $a_j, j = 1, \ldots, N$ *be defined as in (3.18). For* $i \in \{1, \ldots, N\}$ *and* $B \in D$

$$I\!E(S(T_i(B)) = \sum_{n=1}^{N} a_{n \oslash i} \sigma(T_n(B)). \tag{3.19}$$

Proof: In Theorem 10 we replace B by $l_{T_i'}(B)$ and thus obtain

$$\mathbb{E}(S(T_i(B)) = \sum_{j=1}^{N} a_j \sigma(l_{T_j'} l_{T_i'}(B)) = \sum_{j=1}^{N} a_j \sigma(l_{T_i' T_j'}(B)) = \sum_{j=1}^{N} a_j \sigma(T_{i \otimes j}(B)) \ . \tag{3.20}$$

Using Definition 12 we obtain the required result. ∎

Similarly to what we did to obtain Corollary 11 we can extend formula (3.19) by linearity to obtain

Corollary 14 *Let* $a_j = a_j(p) = \frac{p^{m(j)}}{2^{k-m(j)}}$. *For any matrices* $m_1, \ldots, m_k \in M$ *we have*

$$I\!E(r_{\pi_i}(S)(m_1^{\epsilon_1(i)}, \ldots, m_k^{\epsilon_k(i)})) = \sum_{j=1}^{N} a_j r_{\pi_{i \otimes j}}(\sigma)(m_1^{\epsilon_1(i \otimes j)}, \ldots, m_k^{\epsilon_k(i \otimes j)}) \tag{3.21}$$

or, equivalently,

$$I\!E(r_{\pi_i}(S)(m_1^{\epsilon_1(i)}, \ldots, m_k^{\epsilon_k(i)})) = \sum_{n=1}^{N} a_{n \oslash i} r_{\pi_n}(\sigma)(m_1^{\epsilon_1(n)}, \ldots, m_k^{\epsilon_k(n)}).$$

where $(\epsilon(n), \pi_n) = T_n$ *are the particular pairs. The polynomials* $A_{(\pi,\epsilon),(\pi',\epsilon')}$ *in (1.3) are defined by*

$$A_{(\pi_i, \epsilon(i)),(\pi_n, \epsilon(n))}(p) = a_{n \oslash i}(p) = \frac{p^{m_{n \oslash i}}}{2^{k - m_{n \oslash i}}}, \qquad i, n \in \{1, \ldots, N\}. \tag{3.22}$$

If in Corollary 14 we take the m_i as symmetric matrices s_i, the role of the transpose vanishes and we obtain the following corollary.

Corollary 15 *For any symmetric matrices* $s_1, \ldots, s_k \in V$ *we have*

$$I\!E(r_{\pi_i}(S)(s_1, \ldots, s_k)) = \sum_{n=1}^{N} a_{n \oslash i} r_{\pi_n}(\sigma)(s_1, \ldots, s_k). \tag{3.23}$$

Now, since we know that $i \mapsto \pi_i$ is surjective from $\{1, \ldots, N\}$ onto $\mathcal{S}_k$, we can express the polynomials $B_{\pi,\pi'}(p)$ in (1.4) as sums of suitable polynomials $a_i(p)$. Actually, a given permutation $\pi \in \mathcal{S}_k$ appears in $2^{k-m(\pi)}$ different particular pairs, so (3.23) becomes

$$\mathbb{E}(r_{\pi_i}(S)(s_1, \ldots, s_k)) = \sum_{\pi' \in \mathcal{S}_k} B_{\pi_i, \pi'} r_{\pi_n}(\sigma)(s_1, \ldots, s_k).$$

4 Applications of graphs

When we want to apply the formulas of the preceding section to obtain moments of the real Wishart distribution, the main problem which arises is the following: given two particular permutations T'_i and T'_j, how can we identify a particular permutation T'_n such that $T'_i T'_j \sim T'_n$?

In the first part of this section we design a quick graph method allowing us to identify the particular permutation $\tau_\epsilon t_\pi$ equivalent to a given permutation $g \in \mathcal{S}_{2k}$, and we illustrate the application of this method to the problem of the computation of the moments of the Wishart distribution.

In the second part of this section we will develop a second method based on graphs and leading to the practical computation of moments of the real Wishart distribution.

Graph techniques applied to random matrices, permutations and spherical (zonal) polynomials have been fruitfully used before in such works as in [7], [8] and [12]. Recall the notation $j^+ = j + k$.

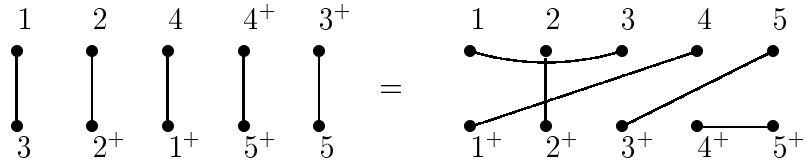
Definition 16 *To a permutation $\pi \in \mathcal{S}_{2k}$ we associate a graph*

$$\eta(\pi) = \{\{\pi(1), \pi(1^+)\}, \{\pi(2), \pi(2^+)\}, \ldots \{\pi(j), \pi(j^+)\}, \ldots, \{\pi(k), \pi(k^+)\}\}.$$

In other words, $\eta(\pi)$ is the graph in which i is adjacent to j if and only if $|\pi^{-1}(i) - \pi^{-1}(j)| = k$.

This definition is analogous to that given in [8] and [12].

Example. Let $k = 5$ and $\pi \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1^+ & 2^+ & 3^+ & 4^+ & 5^+ \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 4^+ & 3^+ \\ 3 & 2^+ & 1^+ & 5^+ & 5 \end{pmatrix}$.
In order to draw the graph $\eta(\pi)$ we connect the elements in the columns:

$\eta(\pi) =$ 

Like in [7], we denote the set of all graphs $\eta(\pi)$ ($\pi \in \mathcal{S}_{2k}$) with vertices $\{1, 2, \ldots, 2k\}$ by $\mathcal{F}_k$.

Proposition 17 *For any $g_1, g_2 \in \mathcal{S}_{2k}$*

$$g_1 H = g_2 H \quad \Leftrightarrow \quad \eta(\pi_1) = \eta(\pi_2).$$

It follows that $\mathcal{S}_{2k}/H \equiv \mathcal{F}_k$.

Proof: This is simply a reformulation of Lemma 3.1(a) in [7] for right cosets, saying that $Hg_1 = Hg_2$ is equivalent to $\eta(g_1^{-1}) = \eta(g_2^{-1})$. In fact $g_1 H = g_2 H$ if and only if $Hg_1^{-1} = Hg_2^{-1}$. ∎

We now show how, given a graph $\eta \in \mathcal{F}_k$, one can easily determine the particular pair (ϵ, π) such that $\eta = \eta(\tau_\epsilon t_\pi)$. The identity graph

$$\eta(e) = \{\{1, 1^+\}, \{2, 2^+\}, \ldots, \{k, k^+\}\}$$

is used in the algorithm given in the following proposition.

Proposition 18 *Let $\eta \in \mathcal{F}_k$. Let γ be the 2-regular graph (i.e. with all its cycles having even length) defined by $\gamma = \eta(e) \cup \eta$. Then $\eta = \eta(\tau_\epsilon t_\pi)$ with $(\epsilon, \pi) \in \mathbf{Z}_2^k \times \mathcal{S}_k$, a particular pair obtained in the following way.*
Consider the even length cycles γ_j, $j = 1, \ldots, J \leq k$, whose disjoint union is equal to γ.

(1) **Identification of π.** *Any cycle γ_j of length $2q_j$ is of the form*

$$\gamma_j = (\overset{+}{o}_{j1}, \overset{o}{+}_{j1}, \ldots, \overset{+}{o}_{jq_j}, \overset{o}{+}_{jq_j}), \quad i_{jl} \in \{1, \ldots, k\} \tag{4.24}$$

where the ordered couple $\overset{+}{o}\overset{o}{+}$ is such that $\overset{+}{o}, \overset{o}{+} \in \{i, i^+\}$ with $\overset{+}{o} = i$ implying $\overset{o}{+} = i^+$ and $\overset{+}{o} = i^+$ implying $\overset{o}{+} = i$.

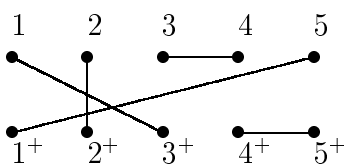
We adopt the convention that the "starting point" is $\overset{+}{o}_{j1} = \min\{i_{jl}^+ \mid l = 1, \ldots, q_j\}$. Set $\pi_j = (i_1, i_2, \ldots, i_{q_j})$. Then $\pi = \prod_{j=1}^J \pi_j$.

(2) **Identification of $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$.** *For a given $m \in \{1, \ldots, k\}$ there exists one and only one i_{jl} in a cycle γ_j as in (4.24) above such that $m = i_{jl}$.*

Then $\epsilon_m = 1$ if the couple $\overset{+}{o}_{jl}\overset{o}{+}_{jl} = i_{jl}^+ i_{jl}$ and $\epsilon_m = -1$ otherwise.

Before proving this proposition, let us give an example to illustrate it.

Example. Let us identify the particular pair corresponding to the following graph $\eta \in \mathcal{F}_k$:

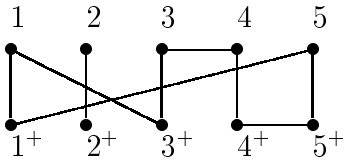
$\eta =$ 

Let us apply the algorithm given in Proposition 18:
$\gamma_1 = (1^+, 1, 3^+, 3, 4, 4^+, 5^+, 5)$, $\gamma_2 = (2^+, 2)$
(1) Identification of π: $\pi_1 = (1, 3, 4, 5)$, $\pi_2 = (2)$, $\pi = (1, 3, 4, 5)(2)$
(2) Identification of ϵ: $\epsilon = (1, 1, 1, -1, 1)$.

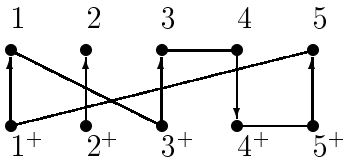
Let us note that the preceding steps may be done very quickly, simply by looking at graphs in the following way.

Add the graph $\eta(e)$ of the identity to the graph η:

$\gamma = \eta(e) \cup \eta =$



We immediately see the two cycles. We start the first cycle at $1^+ = \min\{1^+, 3^+, 4^+, 5^+\}$ and go to 1. We start the second cycle at 2^+. In the picture, this leads to



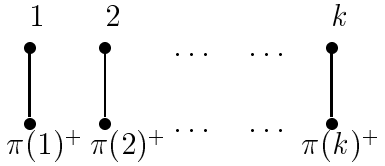
We only put arrows on the vertical edges $\{i, i^+\}$. The corresponding ϵ_i is equal to ± 1 depending on whether the edge $\{i, i^+\}$ is traced in the direction $i^+ \to i$ or $i \to i^+$ respectively (upwards or downwards in the picture).

Therefore $\pi = (1, 3, 4, 5)(2)$ and $\epsilon = (1, 1, 1, -1, 1)$.

Proof: We know by Proposition 17 and Theorem 8 that all the graphs $\eta \in \mathcal{F}_k$ are obtained as graphs of particular permutations.

In the case $\eta = \eta(t_\pi)$ (ie. when $\epsilon = id$), the graph η is of the form

$\eta =$



Applying (1) of the Proposition and starting with an i^+ gives (4.24) equal to

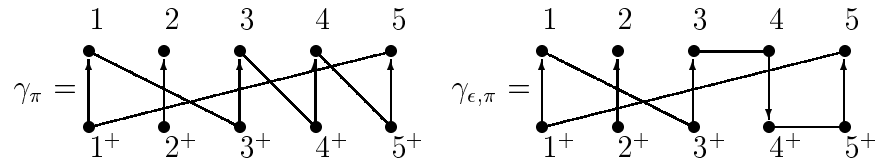
$$(i^+, i, \pi(i)^+, \pi(i), \pi(\pi(i))^+, ...)$$

and finally the cycle $(i, \pi(i), \pi(\pi(i)), ...)$ of π. This is equivalent to the decomposition of π into cycles. It is evident that the edges $\{m, m^+\}$ are all run over in the direction $m^+ \to m$. The proposition is true in this case.

The definition of τ_ϵ implies that the graph of $\tau_\epsilon t_\pi$ is obtained from the graph of t_π by interchanging the vertices m and m^+ whenever $\epsilon_m = -1$.

Consider first the case when only one $\epsilon_m = -1$. The couple m^+m in the decomposition (4.24) of the graph $\gamma_\pi = \eta(e) \cup \eta(t_\pi)$ is replaced by the couple mm^+ in the analogous decomposition of the graph $\gamma_{\epsilon,\pi} = \eta(e) \cup \eta(\tau_\epsilon t_\pi)$. It follows that the cycle π_j that we deduce from (4.24) is the same for γ_π and $\gamma_{\epsilon,\pi}$: *(1)* is therefore proven to be valid and so is *(2)*.

Let us illustrate this part of the proof by an example. As in the last example, take $k = 5$, $\pi = (1, 3, 4, 5)$, $\epsilon = (1, 1, 1, -1, 1)$. Then



In the first case $\gamma_1 = (1^+, 1, 3^+, 3, 4^+, 4, 5^+, 5), \gamma_2 = (2^+, 2)$, $\epsilon = id$. In the second case $\gamma_1 = (1^+, 1, 3^+, 3, 4, 4^+, 5^+, 5)$, $\gamma_2 = (2^+, 2)$, $\epsilon = (1, 1, 1, -1, 1)$. Geometrically, the horizontal edges $\{3, 4\}$ and $\{4^+, 5^+\}$ of the graph $\gamma_{\epsilon,\pi}$ are replaced by the oblique edges $\{3, 4^+\}$ and $\{4, 5^+\}$ of the graph γ_π. In order to pass from $\gamma_{\epsilon,\pi}$ to γ_π one performs a central reflection of the edge $\{4, 4^+\}$.

The general case now follows by decomposing $\epsilon \in \mathbf{Z}_2^k$ into a product of elements with only one entry equal to -1. It is clear that, by construction, (ϵ, π) is a particular pair. ∎

Example. As an application of Proposition 18 let us compute the moments $\mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2Sm_3))$ and $\mathbb{E}(\mathrm{tr}\,(Sm_1Sm_2Sm_3))$ in the case $k = 3$. We apply (3.21) from Corollary 14 and we use the particular pairs listed on page 10. We get

$$\mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2Sm_3)) = p^3\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3)$$
$$+\frac{p^2}{2}\sum_{j=2}^{7} r_{\pi_{2\otimes j}}(\sigma)(m_1^{\epsilon_1(2\otimes j)}, m_2^{\epsilon_2(2\otimes j)}, m_3^{\epsilon_3(2\otimes j)}) + \frac{p}{4}\sum_{j=8}^{15} r_{\pi_{2\otimes j}}(\sigma)(m_1^{\epsilon_1(2\otimes j)}, m_2^{\epsilon_2(2\otimes j)}, m_3^{\epsilon_3(2\otimes j)}). \tag{4.25}$$

As the operation $i \mapsto i \otimes j$ is injective, we only have to compute the values of $2 \otimes j$ for $j = 2, \ldots, 7$, ie. first the compositions of permutations $T_2'T_j' = t_{(23)}\tau_{\epsilon_j}t_{\pi_j}$ and next the equivalent particular permutations $T_n' \sim T_2'T_j'$, using $t_\pi t_{\pi'} = t_{\pi\pi'}$ or, if needed, Proposition 18. For example, $T_2'T_2' = t_{(23)}t_{(23)} = e$, so $2 \otimes 2 = 1$. $T_2'T_3' = t_{(23)}\tau_{(1,1,-1)}t_{(23)} = \begin{pmatrix} 1 & 2 & 2^+ \\ 1 & 3 & 3^+ \end{pmatrix} \sim T_3'$, so $2 \otimes 3 = 3$. In a similar way we find $2 \otimes 4 = 8$, $2 \otimes 5 = 15$, $2 \otimes 6 = 12$ and $2 \otimes 7 = 11$. The results of $2 \otimes 8, \ldots, 2 \otimes 15$ give the set $\{4, 5, 6, 7, 9, 10, 13, 14\}$. Finally

$$\begin{aligned}
&\mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2Sm_3)) = p^3\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3) \\
+\;& \frac{p^2}{2}\Big(\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_3) + \mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3^t) \\
+\;& \mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3) + \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2^t)\Big) \\
+\;& \frac{p}{4}\Big(\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3) + \mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3^t) \\
+\;& \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2) + \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2^t) \\
+\;& \mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3) + \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2)\Big)
\end{aligned} \tag{4.26}$$

Similarly, we compute $8 \otimes 2 = 6,\ 8 \otimes 3 = 9,\ 8 \otimes 4 = 2,\ 8 \otimes 5 = 15,\ 8 \otimes 6 = 4$ and $8 \otimes 7 = 10$ and we deduce the formula

$$\begin{aligned}
&\mathbb{E}(\mathrm{tr}\,(Sm_1Sm_2Sm_3)) = p^3\mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3)\\
+\ &\frac{p^2}{2}\Big(\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3) + \mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3) + \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2)\\
+\ &\mathrm{tr}\,(\sigma m_1\sigma m_2\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2^t)\Big)\\
+\ &\frac{p}{4}\Big(\mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_3) + \mathrm{tr}\,(\sigma m_1)\mathrm{tr}\,(\sigma m_2\sigma m_3^t)\\
+\ &\mathrm{tr}\,(\sigma m_2)\mathrm{tr}\,(\sigma m_1\sigma m_3^t) + \mathrm{tr}\,(\sigma m_3)\mathrm{tr}\,(\sigma m_1\sigma m_2^t)\\
+\ &\mathrm{tr}\,(\sigma m_1\sigma m_2^t\sigma m_3^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2) + \mathrm{tr}\,(\sigma m_1\sigma m_3\sigma m_2^t) + \mathrm{tr}\,(\sigma m_1\sigma m_3^t\sigma m_2)\Big).
\end{aligned} \tag{4.27}$$

It follows that for s_1, s_2, s_3 symmetric, using the additional fact that $\mathrm{tr}\,(\sigma s_1\sigma s_2\sigma s_3) = \mathrm{tr}\,(\sigma s_1\sigma s_3\sigma s_2)$, we have:

$$\begin{aligned}
&\mathbb{E}(\mathrm{tr}\,(Ss_1)\mathrm{tr}\,(Ss_2Ss_3)) = (p^3 + \frac{p^2}{2})\mathrm{tr}\,(\sigma s_1)\mathrm{tr}\,(\sigma s_2\sigma s_3)\\
+\ &\frac{p^2}{2}\mathrm{tr}\,(\sigma s_1)\mathrm{tr}\,(\sigma s_2)\mathrm{tr}\,(\sigma s_3) + (2p^2 + p)\mathrm{tr}\,(\sigma s_1\sigma s_2\sigma s_3)\\
+\ &\frac{p}{2}\Big(\mathrm{tr}\,(\sigma s_2)\mathrm{tr}\,(\sigma s_1\sigma s_3) + \mathrm{tr}\,(\sigma s_3)\mathrm{tr}\,(\sigma s_1\sigma s_2)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}(\mathrm{tr}\,(Ss_1Ss_2Ss_3)) = (p^3 + \frac{3p^2}{2} + p)\mathrm{tr}\,(\sigma s_1\sigma s_2\sigma s_3)\\
+\ &(\frac{p^2}{2} + \frac{p}{4})\Big(\mathrm{tr}\,(\sigma s_1)\mathrm{tr}\,(\sigma s_2\sigma s_3) + \mathrm{tr}\,(\sigma s_2)\mathrm{tr}\,(\sigma s_1\sigma s_3) + \mathrm{tr}\,(\sigma s_3)\mathrm{tr}\,(\sigma s_1\sigma s_2)\Big)\\
+\ &\frac{p}{4}\mathrm{tr}\,(\sigma s_1)\mathrm{tr}\,(\sigma s_2)\mathrm{tr}\,(\sigma s_3).
\end{aligned} \tag{4.28}$$

We see how the polynomials $B_{\pi,\pi'}$ arise as sums of the monomials a_j equal in this case to $p^3, p^2/2$ and $p/4$.

The procedure given above, that we have used to compute quickly and by hand all the moments of the Wishart in the case $k = 3$, can be implemented on the computer using a **MAPLE** program. This program gives the numbers $n = i \otimes j$ and $n \oslash i = j$. For $k = 4$ we have to deal with $N = 105$ particular pairs. When $k = 5$ since the size of A is $N = 945$, we will be content to work with $s_1, \ldots, s_5$ symmetric in order to determine the matrix B.

More graphs' applications. We complete this section by giving another way of using graph techniques in order to simplify and accelerate the computation of moments. This technique avoids the computation of compositions of permutations when applying Corollary 14. To this end we will use the double coset space

$$H\backslash\mathcal{S}_{2k}/H = \{\, HgH \mid g \in \mathcal{S}_{2k}\}.$$

Definition 19 *Let $\eta_1, \eta_2 \in \mathcal{F}_k$. We denote by*

$$\Lambda(\eta_1, \eta_2)$$

the partition of the integer k whose parts are half the cycle-lengths of the graph $\eta_1 \cup \eta_2$.

If $g \in \mathcal{S}_{2k}$ then we write

$$\Lambda(g) = \Lambda(\eta(g), \eta(e)).$$

Let M_n be the set of all partitions of n. For a permutation $g \in \mathcal{S}_n$ denote by $\mathbf{m}(g) \in M_n$ the partition notation of the portrait of g. For example, if $g = 12(3,4)(5,6)(7,8,9,10) \in \mathcal{S}_{10}$ then $\mathbf{m}(g) = (1,1,2,2,4)$ is a partition of $n = 10$ which determines the portrait of g uniquely.

Proposition 20 *(i) The elements $g_1 \in \mathcal{S}_{2k}$ and $g_2 \in \mathcal{S}_{2k}$ belong to the same double class in $H\backslash\mathcal{S}_{2k}/H$ if and only if $\Lambda(g_1) = \Lambda(g_2)$.*

(ii) Let $g, w \in \mathcal{S}_{2k}$. Then $\Lambda(gw^{-1}) = \Lambda(\eta(g), \eta(w))$.

(iii) Let $\pi \in \mathcal{S}_k$ and $\epsilon \in \mathbf{Z}_2^k$. Then $\Lambda(\tau_\epsilon t_\pi) = \mathbf{m}(\pi)$.

Proof: Properties (i) and (ii) go back to James[8], see also [12, page 46] and [7, Lemma 3.1(b)]. Proposition 18 implies (iii). ∎

We now describe another graph method to compute the moments of the real Wishart distribution. This method is based on Proposition 20. The idea is to compute, for each $i \in \{1, \ldots, N\}$, the sets of results $i \otimes j$ (and not all the $i \otimes j$ separately), when T'_j's belong to the same double coset HgH, ie. when $\Lambda(T'_j) = \mathbf{m}$ is fixed. We are looking for n_j's such that $i \otimes j = n_j$ or equivalently such that $T'_i T'_j = g_j$ with $g_j \sim T'_{n_j}$.

Proposition 21 *Let $\mathbf{m} \in M_n$ and $i \in \{1, \ldots, N\}$ be fixed. Then*

$$\Lambda(T'_j) = \mathbf{m} \text{ and } T'_i T'_j = g_j \in \mathcal{S}_{2k} \quad \Leftrightarrow \quad \mathbf{m} = \Lambda(\delta(T'_i), \delta(g_j))$$

where we write $\delta(g) = \eta(g^{-1})$.

Proof: We have $T'_i T'_j = g_j \Leftrightarrow T'_j = T_i'^{-1} g_j$ and when $\Lambda(T'_j) = \mathbf{m}$ is fixed this is equivalent to

$$\mathbf{m} = \Lambda(T'_j) = \Lambda(T_i'^{-1} g_j) = \Lambda(\eta(T_i'^{-1}), \eta(g_j^{-1})) = \Lambda(\delta(T'_i), \delta(g_j)).$$

∎

Remark. Note that $\delta(g)$ is what is called the graph of $g \in \mathcal{S}_{2k}$ in [7]).

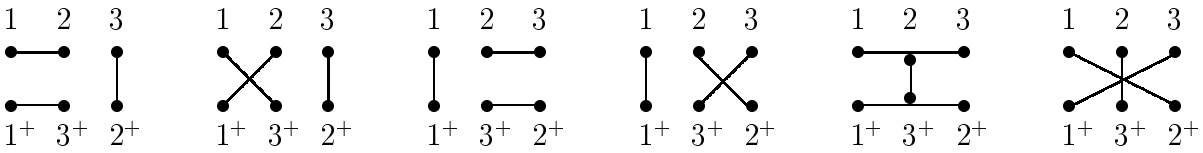
Using graphs it is easy to determine the set of graphs $\delta(g_j)$ such that $\Lambda(\delta(T'_i), \delta(g_j)) = \mathbf{m}$ for i and $\mathbf{m}$ given. Next Proposition 18 may be used in order to determine the n_j's knowing the graphs $\delta(g_j) = \eta(g_j^{-1})$. Moreover one may easily reformulate Proposition 18 in order to determine the particular pair (ϵ, π) such that $\tau_\epsilon t_\pi \sim g$ knowing $\delta(g) = \eta(g^{-1})$. In part (1) of Proposition 18 just change the direction of running over the cycles by starting with $i \to i^+$ and in part (2) put $\epsilon_m = 1$ if the couple $i_{jl}^{\overset{+}{o}} i_{jl}^{\overset{o}{+}} = i_{jl} i_{jl}^+$

and $\epsilon_m = -1$ otherwise.

Let us illustrate this method for $k = 3$ and the computation of $\mathbb{E}(\mathrm{tr}\,(Sm_1)\mathrm{tr}\,(Sm_2Sm_3))$. In formula (4.25) the coefficients $j = 2, \ldots, 7$ correspond to $\Lambda(T'_j) = (2,1) \in M_3$. We have $T'_2 = t_{(23)} = T_2'^{-1}$ and the graph $\delta(T'_2) = \eta(T'_2)$ is

$$\delta(T'_2) = \quad \begin{array}{ccc} 1 & 2 & 3 \\ \bullet & \bullet & \bullet \\ | & | & | \\ \bullet & \bullet & \bullet \\ 1^+ & 3^+ & 2^+ \end{array}$$

The six graphs $\delta(g_j)$ are



Next we quickly identify the particular permutations equivalent to the g_j's as $T'_{15}, T'_8, T'_3, T'_1, T'_{11}, T'_{12}$ respectively. As before, we deduce that the results of $2 \otimes 8, \ldots, 2 \otimes 15$ give the set $\{4,5,6,7,9,10,13,14\}$ and we obtain the formula (4.26).

When $k = 4$ the number of portraits is 5 and the computation of a moment formula boils down to the identification of 56 graphs, 56 being the number of permutations with portrait $\mathbf{m} = (2,1,1), (2,2)$ or $(3,1)$. Let us write two of these formulas, for s_1, s_2, s_3, s_4 symmetric (the "full" formulas for m_1, m_2, m_3, m_4 have 105 terms). We use the fact that $r_\pi(\sigma)(s_1, \ldots, s_k) = r_{\pi^{-1}}(\sigma)(s_1, \ldots, s_k)$ when $s_1, \ldots, s_k$ are symmetric and we write $r_\pi(\sigma)(s_1, \ldots, s_k) = r_\pi$ for short.

$$\begin{aligned}
&\mathbb{E}(\mathrm{tr}\,(Ss_1)\mathrm{tr}\,(Ss_2)\mathrm{tr}\,(Ss_3Ss_4)) = (p^4 + \frac{p^3}{2})r_{12(34)} \\
&+\frac{p^3}{2}r_{id} + (p^3 + \frac{p^2}{2})r_{(12)(34)} + (2p^3 + p^2)(r_{1(234)} + r_{2(134)}) \\
&+\frac{p^2}{2}(r_{(12)34} + r_{14(23)} + r_{13(24)} + r_{23(14)} + r_{24(13)}) + 2p^2 r_{(1324)} + 4p^2 r_{(1342)} \\
&+\frac{p}{2}(r_{(13)(24)} + r_{(14)(23)}) + p(r_{3(124)} + r_{4(123)}) + 3p r_{(1234)}
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}(\mathrm{tr}\,(Ss_1Ss_2Ss_3Ss_4)) = (p^4 + 2p^3 + \frac{7}{4}p^2 + \frac{1}{2}p)r_{(1234)} + (\frac{p^3}{2} + \frac{p^2}{2} + \frac{p}{8})(r_{(12)(34)} + r_{(14)(23)}) \\
&+(\frac{p^3}{2} + \frac{3}{4}p^2 + \frac{p}{2})(r_{1(234)} + r_{2(134)} + r_{3(124)} + r_{4(123)}) + (\frac{p^3}{2} + \frac{7}{4}p^2 + p)(r_{(1324)} + r_{(1324)}) \\
&+(\frac{p^2}{4} + \frac{3p}{8})r_{(13)(24)} + (\frac{p^2}{4} + \frac{p}{8})(r_{12(34)} + r_{13(24)} + r_{14(23)} + \ldots + r_{34(12)}) + \frac{p}{8}r_{id}.
\end{aligned}$$

5 Applications of groups and representations

In Corollary 14 we have seen that the matrix $(A_{ij})_{i,j=1,\ldots,N}$ containing the coefficients of all the formulas for the moments of a Wishart distribution on real symmetric matrices, where

$$A_{ij} = A_{(\pi_i,\epsilon(i)),(\pi_j,\epsilon(j))}(p)$$

is determined by (3.22), ie.

$$A_{ij}(p) = a_{j\oslash i}(p) = \frac{p^{m_{j\oslash i}}}{2^{k-m_{j\oslash i}}}, \qquad i,j \in \{1,\ldots,N\}. \tag{5.29}$$

(recall that $N = |S_{2k}/H| = 1 \times 3 \times \ldots (2k-1)$.)

Our goal in this section is to present a mathematical interpretation of the matrix A and to explain in which way the representation theory of the symmetric group may be used in order to determine A.

Definition 22 *Let $\lambda \in M_k$ be a partition of k. We define a square $N \times N$ matrix E_λ in the following way:*

$$E_\lambda = (e_{ij})$$

where $e_{ij} = 1$ if $\mathbf{m}(\pi_{j\oslash i}) = \lambda$ and $e_{ij} = 0$ otherwise.

Note that $E_{1,1,\ldots,1} = \mathrm{Id}$, the identity matrix. We also observe that

$$\sum_{\lambda \in M_k} E_\lambda = \mathbf{1}$$

where $\mathbf{1}$ denotes a $N \times N$ matrix with all coefficients equal to 1.

Proposition 23 *The matrices E_λ are symmetric.*

Proof: Fix $\lambda \in M_k$. By Propositions 18 (1) and 20 and using the fact that $\Lambda(\pi) = \Lambda(\pi^{-1})$ we have $e_{ij} = 1 \Leftrightarrow \Lambda(T_i'^{-1}T_j') = \lambda \Leftrightarrow \Lambda(T_j'^{-1}T_i') = \lambda \Leftrightarrow e_{ji} = 1$. ∎

For $\lambda \in M_k$ let us denote by m_λ the number of non-zero terms of the partition λ. If a permutation $\pi \in \mathcal{S}_k$ has a portrait indexed by the partition λ then the number of cycles is $m(\pi) = m_\lambda$. Note that the sum of coefficients in any column of the matrix E_λ is equal to the number n_λ of particular pairs (ϵ, π) with $\mathbf{m}(\pi) = \lambda$.

The following theorem follows from (5.29).

Theorem 24 *The matrix A can be represented as the following weighted sum of matrices E_λ:*

$$A = \sum_{\lambda \in M_k} \frac{p^{m_\lambda}}{2^{k-m_\lambda}} E_\lambda \ . \tag{5.30}$$

In particular, the matrix A is symmetric.

For example, if $k=3$, then the 15×15 matrix A is equal to

$$A = p^3\,\mathrm{Id} + \frac{p^2}{2}\,E_{2,1} + \frac{p}{4}\,E_3$$

and if $k=4$, then the 105×105 matrix A decomposes as

$$A = p^4\,\mathrm{Id} + \frac{p^3}{2}\,E_{2,1,1} + \frac{p^2}{4}\,(E_{2,2}+E_{3,1}) + \frac{p}{8}E_4.$$

Thus, in order to find the matrix A it is sufficient to determine the matrices E_λ. The graph method based on Propositions 20 and 21 in Section 4 may be used to compute the E_λ's. When $k=3$ we obtain

$$E_{2,1} = \begin{pmatrix}
0&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&1&0&0&1&1&0&0&1\\
1&1&0&0&0&0&0&0&1&1&0&0&1&1&0\\
1&0&0&0&1&0&0&1&0&1&0&1&0&1&0\\
1&0&0&1&0&0&0&0&1&0&1&0&1&0&1\\
1&0&0&0&0&0&1&1&1&0&0&1&1&0&0\\
1&0&0&0&0&1&0&0&0&1&1&0&0&1&1\\
0&1&0&1&0&1&0&0&1&1&0&0&0&0&1\\
0&0&1&0&1&1&0&1&0&0&1&0&0&1&0\\
0&0&1&1&0&0&1&1&0&0&1&0&1&0&0\\
0&1&0&0&1&0&1&0&1&1&0&1&0&0&0\\
0&1&0&1&0&1&0&0&0&0&1&0&1&1&0\\
0&0&1&0&1&1&0&0&0&1&0&1&0&0&1\\
0&0&1&1&0&0&1&0&1&0&0&1&0&0&1\\
0&1&0&0&1&0&1&1&0&0&0&0&1&1&0
\end{pmatrix} \tag{5.31}$$

and we easily obtain $A = p^3\,\mathrm{Id} + \frac{p^2}{2}\,E_{2,1} + \frac{p}{4}\,(\mathbf{1} - \mathrm{Id} - E_{2,1})$.

Kates([12]) used an analogous graph technique in the double coset space $H\backslash\mathcal{S}_{2k}/H$ setting, in order to compute the spherical functions of the Gelfand pair $(\mathcal{S}_{2k}, H)$ and to obtain in this way the coefficients of zonal polynomials. Let the $p(k)\times p(k)$ matrices C_λ(where $p(k)$ is the number of different partitions of k) be the following "contractions" of the matrices E_λ, passing from one-sided cosets gH to the double cosets HgH:
in order to compute the coefficient $c_{\mu\nu}$ of C_λ $(\mu,\nu\in M_k)$ choose any column of E_λ, say the j-th, with $\mathbf{m}(\pi_j)=\nu$ and sum the coefficients of this column in the lines indexed by the i's such that $\mathbf{m}(\pi_i)=\nu$.

The eigenvectors of C_λ are the spherical functions on $(\mathcal{S}_{2k}, H)$ and Kates([12]) constructed the matrices C_λ using graph techniques. Observe also that if we call $\tilde{A}$ the $p(k)\times p(k)$ matrix obtained from the matrix A by the above "contraction" then

$$\mathbb{E}(r_\mu(S)(m,\dots,m)) = \sum_{\nu\in M_k} \tilde{a}_{\mu\nu} r_\nu(\sigma)(m,\dots,m)$$

where we write r_μ instead of any r_π when π has portait μ. The matrix $\tilde{A} = \sum_{\lambda\in M_k} \frac{p^{m_\lambda}}{2^{k-m_\lambda}} C_\lambda$ may be independently deduced from the results of [13]. We may understand the matrix A as an unfolding of the matrix $\tilde{A}$.

By an analogous convenient contraction of the matrix A we may compute the matrix B defined in (1.4). For $k = 3$ we may of course just use Proposition 1 and the formulas (4.26), (4.27), (4.28) to obtain

$$B = \begin{pmatrix} p^3 & p^2 & p^2 & p^2 & p & p \\ \frac{p^2}{2} & p^3 + \frac{p^2}{2} & \frac{p}{2} & \frac{p}{2} & p^2 + \frac{p}{2} & p^2 + \frac{p}{2} \\ \frac{p^2}{2} & \frac{p}{2} & p^3 + \frac{p^2}{2} & \frac{p}{2} & p^2 + \frac{p}{2} & p^2 + \frac{p}{2} \\ \frac{p^2}{2} & \frac{p}{2} & \frac{p}{2} & p^3 + \frac{p^2}{2} & p^2 + \frac{p}{2} & p^2 + \frac{p}{2} \\ \frac{p}{4} & \frac{p}{4} + \frac{p^2}{2} & \frac{p}{4} + \frac{p^2}{2} & \frac{p}{4} + \frac{p^2}{2} & p^3 + p^2 + \frac{p}{4} & \frac{p^2}{2} + \frac{3p}{4} \\ \frac{p}{4} & \frac{p}{4} + \frac{p^2}{2} & \frac{p}{4} + \frac{p^2}{2} & \frac{p}{4} + \frac{p^2}{2} & \frac{p^2}{2} + \frac{3p}{4} & p^3 + p^2 + \frac{p}{4} \end{pmatrix}$$

where the permutations π in the vectors $[\mathbb{E}(r_\pi(S))]$ and $[r_\pi(\sigma)]$ are ordered as follows: $id,\ (23),\ (13),\ (12),\ (123),\ (132)$.

We now move to the main subject of this section, that is the discussion of applications of group and representation theory to the matrices E_λ and A. Let $\mathcal{A}(\mathcal{S}_{2k})$ be the algebra of $\mathcal{S}_{2k}$, ie. the real linear space of functions from $\mathcal{S}_{2k}$ to $\mathbb{R}$. Define the convolution of functions on $\mathcal{S}_{2k}$ by

$$f * g(x) = \sum_{y \in \mathcal{S}_{2k}} f(xy^{-1})g(y) = \sum_{y \in \mathcal{S}_{2k}} f(y)g(y^{-1}x)$$

and let $R(g)$ denote the (right) convolution operator $f \mapsto f * g$ on the group algebra $\mathcal{A}(\mathcal{S}_{2k})$. Suppose that g is a right H-invariant function on $\mathcal{S}_{2k}$. Then

$$R(g) : \mathcal{A}(\mathcal{S}_{2k}) \ \rightarrow \ \mathcal{A}(\mathcal{S}_{2k}/H)$$

where we denote by $\mathcal{A}(\mathcal{S}_{2k}/H)$ the space of right H-invariant functions on $\mathcal{S}_{2k}$. It follows that if ϕ is an eigenvector of $R(g)$ corresponding to a non-zero eigenvalue then ϕ is right H-invariant.

Let $\lambda \in M_k$. Denote by $\delta_\lambda \in \mathcal{A}(\mathcal{S}_{2k})$ the indicator function of the double coset $D_\lambda = \{w \in \mathcal{S}_{2k} \mid \Lambda(w) = \lambda\} \in H\backslash\mathcal{S}_{2k}/H$ (see Proposition 20(i)). We have $\delta_\lambda \in \mathcal{A}(H\backslash\mathcal{S}_{2k}/H)$. We write $\delta'_\lambda = \frac{1}{|H|}\delta_\lambda$.

Similarly, denote by $\delta_{C_i} \in \mathcal{A}(\mathcal{S}_{2k}/H)$ the indicator function of a coset $C_i \in \mathcal{S}_{2k}/H$, $i = 1, \ldots, N$, and by $\delta'_{C_i} = \frac{1}{|H|}\delta'_{C_i}$ the normalized indicator function (observe that $|C_i| = |H|$.) The algebra $\mathcal{A}(\mathcal{S}_{2k}/H)$ can be identified with $\mathbb{R}^N$ and $\{\delta'_{C_i}\}_{i=1,\ldots,N}$ is a canonical linear basis of $\mathcal{A}(\mathcal{S}_{2k}/H)$.

Proposition 25 *Let $\mathcal{R}_\lambda = R(\delta'_\lambda)|_{\mathcal{A}(\mathcal{S}_{2k}/H)}$. The matrix of the operator*

$$\mathcal{R}_\lambda : \mathcal{A}(\mathcal{S}_{2k}/H) \rightarrow \mathcal{A}(\mathcal{S}_{2k}/H)$$

in the basis $\{\delta'_{C_i}\}_{i=1,\ldots,N}$ is equal to the matrix E_λ ($\lambda \in M_k$).

Proof: In order to determine the coefficient m_{ij} of the matrix of the operator $\mathcal{R}_\lambda$ on $\mathcal{A}(\mathcal{S}_{2k}/H)$ we write

$$\mathcal{R}_\lambda(\delta'_{C_j}) = \sum_l m_{lj}\delta'_{C_l}$$

and we observe that

$$\sum_{z\in C_i} \mathcal{R}_\lambda(\delta'_{C_j})(z) = m_{ij}$$

which means that

$$\sum_{z\in C_i} \delta_{C_j} * \delta_{H g_\lambda H}(z) = |H|^2 m_{ij}$$

where g_λ is any element of the double coset D_λ.

Let $T_i = (\epsilon_i, \pi_i) \in \mathbf{Z}_2^k \times \mathcal{S}_k$ be a particular pair such that the particular permutation $T'_i = \tau_{\epsilon_i} t_{\pi_i} \in C_i$. Then $C_i = T'_i H$ and the left-hand side of the previous equality becomes

$$\begin{aligned}
\sum_{z\in C_i} \delta_{C_j} * \delta_{H g_\lambda H}(z) &= |\{(x,y) \in T'_j H \times H g_\lambda H \mid xy = z \in T'_i H\}| \\
&= |\{(x,z) \in T'_j H \times T'_i H \mid x^{-1}z = y \in H g_\lambda H\}| \\
&= |H| \times |\{z \in T'_i H \mid T'^{-1}_j z \in H g_\lambda H\}| \\
&= |H| \times |\{z \in T'_i H \mid z^{-1} T'_j \in H g_\lambda H\}|.
\end{aligned}$$

Now, if $T'^{-1}_i T'_j \in H g_\lambda H$ then

$$|\{z \in T'_i H \mid z^{-1} T'_j \in H g_\lambda H\}| = |H|$$

because all the elements $z = T'_i h$, $h \in H$, verify $z^{-1} T'_j \in H g_\lambda H$.

For the same reason, if $T'^{-1}_i T'_j \notin H g_\lambda H$ then

$$|\{z \in T'_i H \mid z^{-1} T'_j \in H g_\lambda H\}| = 0.$$

This implies that $m_{ij} = 1$ if and only if $T'^{-1}_i T'_j \in D_\lambda$ which is equivalent, by Propositions 18 (1) and 20 to $j = i \otimes l$ with $\Lambda(T'_l) = \Lambda(T'_{j \oslash i}) = \mathbf{m}(\pi_{j\oslash i}) = \lambda$. ∎

Remarks. 1. Kates([12, p.44-46]) has shown in a similar way that the matrix of the operator $R(\delta'_\lambda)|_{\mathcal{A}(H\backslash\mathcal{S}_{2k}/H)} : \mathcal{A}(H\backslash\mathcal{S}_{2k}/H) \to \mathcal{A}(H\backslash\mathcal{S}_{2k}/H)$ in the basis $\{\frac{1}{|D_\lambda|}\delta_\lambda\}$ is equal to C_λ.

2. It is easy to verify that the operators $\mathcal{R}_\lambda$ are symmetric (ie. $\langle \mathcal{R}_\lambda f, g\rangle = \langle f, \mathcal{R}_\lambda g\rangle$, for all $f \in \mathcal{A}(\mathcal{S}_{2k}/H)$). This gives another proof of Proposition 23.

Let us summarize Theorem 24 and Proposition 25 into the following.

Corollary 26 *The matrix $A(p)$ is a linear combination of matrices E_λ of right convolution operators with H-biinvariant functions $\frac{1}{|H|}\delta_\lambda$, in the canonical basis $\{\frac{1}{|H|}\delta_{C_i}\}$ on $\mathcal{A}(\mathcal{S}_{2k}/H)$:*

$$A = \sum_{\lambda\in M_k} a(\lambda) E_\lambda, \quad \text{with } a(\lambda) = p^{m_\lambda}/2^{k-m_\lambda}.$$

The symmetric matrices $(E_\lambda)_{\lambda\in M_k}$ and $(A(p))_{p>0}$ commute. In particular, they have common eigenvectors.

Proof: The functions δ_λ are H-biinvariant and $(\mathcal{S}_{2k}, H)$ is a Gelfand pair, so the operators $R(\delta_\lambda)$ commute and so do the $\mathcal{R}_\lambda$. ∎

A new method of computation of the matrices E_λ and A may be deduced from Corollary 26. The coefficients of the matrix E_λ may be computed as

$$e_{ij} = \frac{1}{|H|}\delta_{C_j} * \delta_{Hg_\lambda H}(T_i').$$

This formula can be programmed on a computer in a simple way.

Our objective now is to determine the eigenvectors and the eigenvalues of the matrix A and thus, to provide an alternative method for its construction.

Note that the functions $f = \delta_\lambda$ are not central. Consider for instance $k = 3$ and two permutations $\pi_1 = (123)(456)$ and $\pi_2 = (145)(236)$ from $\mathcal{S}_{2k}$. Then $\Lambda(\pi_1) = (1, 1, 1)$ (π_1 is in the double coset of the identity) whereas $\Lambda(\pi_2) = (3)$. We have $\delta_{(3)}(\pi_1) = 0 \neq 1 = \delta_{(3)}(\pi_2)$ but π_1 and π_2 have the same portrait $(3, 3)$.

Knowing the character tables of the group $\mathcal{S}_{2k}$ is not sufficient for our problem. A dimension argument confirms it: for example, for $k = 3$ we have 11 characters on $\mathcal{S}_{2k}$ but the matrix A has $N = 15$ eigenvectors.

The functions $f = \delta_\lambda'$ form a basis of $\mathcal{A}(H\backslash\mathcal{S}_{2k}/H)$, so the eigenvectors of the matrices E_λ are the coefficients of the eigenfunctions of all convolution operators $R(f)$, $f \in \mathcal{A}(H\backslash\mathcal{S}_{2k}/H)$, in the basis $\{\delta_{C_i}'\}$. We need some information from representation theory, taking into account the fact that $(\mathcal{S}_{2k}, H)$ is a Gelfand pair(see e.g. [1], [18]).

Let (π, V_π, u_π) be an irreducible H-spherical representation of $\mathcal{S}_{2k}$. This means that there exists a vector $u_\pi \in V_\pi$, $u_\pi \neq 0$, which is invariant by H: $\pi(h)u_\pi = u_\pi$ for all $h \in H$. Such a vector is called spherical and is unique up to a scalar factor. We also know that u_π is cyclic, i.e. the vector space generated by its orbit $\pi(\mathcal{S}_{2k})u_\pi$ is equal to V_π. Denote by $(\cdot, \cdot)$ the scalar product on V_π such that all the operators $\pi(g)$ are unitary on V_π.

Denote by $\hat{\mathcal{S}}_{2k}^H$ the set of all irreducible H-spherical representations of $\mathcal{S}_{2k}$.

Theorem 27 *All eigenfunctions of the operators $R(f)$, $f \in \mathcal{A}(H\backslash\mathcal{S}_{2k}/H)$ are obtained as the functions*

$$\phi_v^\pi(g) = (\pi(g^{-1})v\ ,\ u_\pi), \quad v \in V_\pi$$

where (π, V_π, u_π) is an irreducible H-spherical representation of $\mathcal{S}_{2k}$. The functions ϕ_v^π are right H-invariant.

We have $R(f)\phi_v^\pi = \phi_v^\pi * f = \hat{f}(\pi)\phi_v$ and the eigenvalue $\hat{f}(\pi)$ is given by

$$\hat{f}(\pi) = \frac{1}{|\mathcal{S}_{2k}|}\sum_{g\in\mathcal{S}_{2k}} \varphi^\pi(g^{-1})f(g),$$

where $\varphi^\pi(g) = \frac{1}{\|u_\pi\|^2}(\pi(g^{-1})u_\pi\ ,\ u_\pi)$ is the H-biinvariant spherical function induced by π. In particular, all the eigenfunctions ϕ_v^π of $R(f)$ correspond to the same eigenvalue $\hat{f}(\pi)$.

Note that if $\{v_i\}_{i=1,\dots,d_\pi}$ is a basis of V_π(we write $\dim V_\pi = d_\pi$ for the dimension of the representation π) then the fact that u_π is cyclic implies that the eigenfunctions $\phi^\pi_{v_i}$ are independent. On the other hand any function ϕ^π_v is a linear combination of the $\phi^\pi_{v_i}$'s so that each irreducible spherical representation π gives rise to d_π independent common eigenfunctions of the operators $\mathcal{R}_\lambda$.

The decomposition of the regular representation of $\mathcal{S}_{2k}$ on $\mathcal{A}(\mathcal{S}_{2k})$ into orthogonal irreducible representations guarantees that the set of eigenvectors coming from different representations form an independent family. In particular

$$N = \sum_{\pi \in \hat{\mathcal{S}}^H_{2k}} d_\pi. \tag{5.32}$$

For $f \in \mathcal{A}(\mathcal{S}_{2k}/H)$ let $c(f) = [c_1, \dots, c_N]^\top$ be the vector of coefficients of f in the basis δ'_{C_i}:

$$f = \sum_{i=1}^{N} c_i \, \delta'_{C_i}.$$

The following theorem is the main result of this section. It can be deduced from Theorem 27 together with subsequent remarks and Corollary 26.

Theorem 28 *The matrix A has N independent eigenvectors*

$$c(\phi^\pi_v), \quad \pi \in \hat{\mathcal{S}}^H_{2k}, \quad v \in B_\pi,$$

where B_π is a linear basis of V_π.

The eigenvalue of A corresponding to the vector $c(\phi^\pi_v)$ is equal to

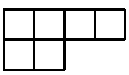
$$e_\pi = \sum_{\lambda \in M_k} a(\lambda) \hat{\delta}'_\lambda(\pi), \quad \pi \in \hat{\mathcal{S}}^H_{2k}$$

where $a(\lambda) = p^{m(\lambda)}/2^{k-m(\lambda)}$.

We will see that the eigenvalues e_π are non zero and pairwise distinct.

Applications of the theory of representations of the symmetric group. In order to go further with our representation-theoretic approach of the matrices A and E_λ and to give examples we will now use some information on the construction of the irreducible representations of $\mathcal{S}_k$. On this topic, the reader is referred to classics such as [9], [16] and also to other less classical but very readable books such as [5], [15] and [17].

We recall that all irreducible representations of $\mathcal{S}_k$ are indexed by the partitions $\mathbf{m} \in M_k$. To each partition $\mathbf{m}$ corresponds a Young diagram. For example, the Young diagram corresponding to $\mathbf{m} = (4, 2)$ is the following



A permutation of $\{1, \ldots, k\}$ combined with such a diagram gives a Young tableau. For example

6	4	1	3
5	2		

If t is a Young tableau, denote by R_t the isotropy group of its lines and by C_t the isotropy group of its columns. For example, if t is as above, then $R_t = \mathcal{S}_{\{1,3,4,6\}} \times \mathcal{S}_{\{2,5\}}$ and $C_t = \mathcal{S}_{\{5,6\}} \times \mathcal{S}_{\{2,4\}}$. The equivalence classes of tableaux with respect to the equivalence relation

$$t_1 \sim t_2 \quad \Leftrightarrow \pi t_1 = t_2, \;\; \pi \in R_t$$

are called *tabloids*. In other words, two tableaux correspond to the same tabloid if and only if the elements of their lines are identical. For example, the tableau above belongs to the tabloid

1	3	4	6
2	5		

We denote by $\{t\}$ the tabloid of t. Let $\mathcal{T}_{\mathbf{m}}$ be the set of all tabloids corresponding to the Young diagram of $\mathbf{m}$. The symmetric group acts on $\mathcal{T}_{\mathbf{m}}$ and $\pi\{t\} = \{\pi t\}$ for $\pi \in \mathcal{S}_k$. We still denote by $\mathcal{A}(\mathcal{T}_{\mathbf{m}})$ the vector space of all real functions on $\mathcal{T}_{\mathbf{m}}$. Of course we may embed $\mathcal{A}(\mathcal{T}_{\mathbf{m}}) \subset \mathcal{A}(\mathcal{S}_k)$. The irreducible representation $\pi_{\mathbf{m}}$ of $\mathcal{S}_k$ is the (left) regular representation of $\mathcal{S}_k$ restrained to the subspace $F_{\mathbf{m}} \subset \mathcal{A}(\mathcal{T}_{\mathbf{m}}) \subset \mathcal{A}(\mathcal{S}_k)$ defined as the linear span of all the functions

$$e_t = \sum_{\pi \in C_t} \epsilon_\pi \, \delta_{\pi\{t\}}$$

where t is any Young tableau corresponding to $\mathbf{m}$ and $\delta_{\pi\{t\}} \in \mathcal{A}(\mathcal{T}_{\mathbf{m}})$ is the indicator function of the tabloid $\pi\{t\}$. The functions e_t are sometimes called *polytabloids* and the space $F_{\mathbf{m}}$ the *Specht module* corresponding to the partition $\mathbf{m}$. A linear basis of the irreducible space $F_{\mathbf{m}}$ is given by so called standard polytabloids e_t, i.e. the functions e_t corresponding to the tableaux t with elements increasing in all lines and columns of the diagram.

Let us illustrate all these concepts on our example where $k = 6$ and $\mathbf{m} = (4, 2)$. There are 15 tabloids in $\mathcal{T}_{(4,2)}$ (we may identify them with the elements of the second line of the Young diagram of $(4, 2)$ and one may choose 2 elements among 6 in 15 different ways). We will write δ_{ij} for the indicator function of the tabloid containing i and j in the second line. The standard tableaux are the following:

1	3	5	6
2	4		

1	3	4	6
2	5		

1	3	4	5
2	6		

1	2	5	6
3	4		

1	2	4	6
3	5		

1	2	4	5
3	6		

1	2	3	6
4	5		

1	2	3	5
4	6		

1	2	3	4
5	6		

Let us denote these standard tableaux by $t_{24}, \ldots, t_{56}$ and the corresponding polytabloid functions by $e_{ij} = e_{t_{ij}}$. In order to compute e_{24} we remark that the column isotropy group $C_{t_{24}} = \mathcal{S}_{\{1,2\}} \times \mathcal{S}_{\{3,4\}}$ has 4 elements. The definition of e_t then yields $e_{24} =$

$\delta_{24} - \delta_{14} - \delta_{23} + \delta_{13}$. This way, we can compute the standard linear basis of the Specht module $F_{(4,2)}$:

$$e_{24} = \delta_{24} - \delta_{14} - \delta_{23} + \delta_{13}, \quad e_{25} = \delta_{25} - \delta_{15} - \delta_{23} + \delta_{13}, \quad e_{26} = \delta_{26} - \delta_{16} - \delta_{23} + \delta_{13},$$
$$e_{34} = \delta_{34} - \delta_{14} - \delta_{23} + \delta_{12}, \quad e_{35} = \delta_{35} - \delta_{15} - \delta_{23} + \delta_{12}, \quad e_{36} = \delta_{36} - \delta_{16} - \delta_{23} + \delta_{12},$$
$$e_{45} = \delta_{45} - \delta_{15} - \delta_{24} + \delta_{12}, \quad e_{46} = \delta_{46} - \delta_{16} - \delta_{24} + \delta_{12}, \quad e_{56} = \delta_{56} - \delta_{16} - \delta_{25} + \delta_{12}.$$

Let us come back to the study of the H-spherical representations of $\mathcal{S}_{2k}$.

Theorem 29 *The H-spherical representations of the symmetric group $\mathcal{S}_{2k}$ are exactly the representations $\pi_{\mathbf{m}} \in \hat{\mathcal{S}}_{2k}$ such that*

$$\frac{1}{2}\mathbf{m} \in M_k.$$

Equivalently, all terms of the partition $\mathbf{m}$ are even numbers. In particular, $|\hat{\mathcal{S}}_{2k}^H| = p(k)$.

Proof: This theorem is known and used in the statistical literature (see e.g.[12]). It may be proved using for example the results in [3], Ch.12.10 (in particular Lemma 12.10.6) but we will give here an elementary constructive proof of the sufficiency of the condition $\frac{1}{2}\mathbf{m} \in M_k$. We will construct a spherical vector for the representation of $\mathcal{S}_{2k}$ corresponding to a fixed partition $\mathbf{m}$ of $2k$ with all terms even.

Consider the set ω of $k!$ tableaux $\mathbf{t}_\pi$ obtained by filling up the Young diagram of $\mathbf{m}$ with the ordered pairs $\{j, j^+\}$, $j = 1, \ldots, k$, in all possible ways according to all permutations $\pi \in \mathcal{S}_k$, line after line, from left to right and from the top to the bottom of the diagram.

Denote by $\mathcal{T}_o$ the set of tabloids corresponding to tableaux $\mathbf{t} \in \omega$.

The group H is generated by the transpositions (j, j^+) and by the permutations s_σ, $\sigma \in \mathcal{S}_k$. The action of (j, j^+) on a tabloid $\{\mathbf{t}\} \in \mathcal{T}_o$ leaves it unchanged. The action of s_σ:

$$s_\sigma(\{\mathbf{t}_\pi\}) = \{\mathbf{t}_{\sigma\pi}\}$$

can change the tabloid, but the image of $\mathcal{T}_o$ under this action is equal to $\mathcal{T}_o$. It follows that the indicator function $\delta_{\mathcal{T}_o} \in \mathcal{A}(\mathcal{T}_{\mathbf{m}})$ is an H-invariant function:

$$\pi(h)\delta_{\mathcal{T}_o} = \delta_{\mathcal{T}_o}, \quad \forall h \in H.$$

Let $\mathbf{t}_o \in \omega$ be the tableau corresponding to $\pi = e$ (identity). It is easy to see that the scalar product

$$(e_{\mathbf{t}_o}, \delta_{\mathcal{T}_o}) \neq 0. \tag{5.33}$$

Actually, if the indicator function $\delta_{\{\mathbf{t}\}}$ of an element $\{\mathbf{t}\} \in \mathcal{T}_o$, $\mathbf{t} \in \omega$, appears in the sum defining $e_{\mathbf{t}_o}$ then the corresponding permutation $\pi \in C_{\mathbf{t}_o}$ has positive signature (π must permute the pairs $\{j, j^+\}$).

Let u be the orthogonal projection of the function $\delta_{\mathcal{T}_o}$ on the Specht module $F_{\mathbf{m}}$. It follows from (5.33) that $u \neq 0$.

If the orthogonal decomposition of $\delta_{\mathcal{T}_o}$ is $\delta_{\mathcal{T}_o} = w+u$, $w \in F_{\mathbf{m}}^{\perp}$, then $\langle \pi(h)w, \pi(h)u \rangle = 0$. Since $u \in F_{\mathbf{m}}$, a representation space for $\mathcal{S}_{2k}$, we have $\pi(h)u \in F_{\mathbf{m}}$. Now the H-invariance of $\delta_{\mathcal{T}_o}$ and the unicity of orthogonal decomposition imply that $\pi(h)u = u$ for all $h \in H$. Thus the irreducible representation $\pi_{\mathbf{m}}$ of $\mathcal{S}_{2k}$ on $F_{\mathbf{m}}$ is H-spherical.

To see that the condition $\frac{1}{2}\mathbf{m} \in M_k$ is necessary, we can use a dimensionality argument (see [3]; cf. (5.32)). ∎

Using Theorems 28 and 29 we see that the matrix A may have at most $p(k)$ different eigenvalues $e_{\pi_{2\mathbf{m}}}$, $\mathbf{m} \in M_k$. We may say more about these eigenvalues. Recall the definition of the gamma function of a real symmetric cone Ω:

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{\frac{r(r-1)}{4}} \prod_{j=1}^{r} \Gamma(s_j - \frac{1}{2}(j-1)).$$

Theorem 30 *The eigenvalues of the matrix A are given by*

$$e_{\pi_{2\mathbf{m}}} = \frac{\Gamma_\Omega(p+\mathbf{m})}{\Gamma_\Omega(p)}, \quad \mathbf{m} \in M_k.$$

In particular the matrix A has exactly $p(k)$ eigenvalues which are all different from zero when $\frac{\Gamma_\Omega(p+\mathbf{m})}{\Gamma_\Omega(p)} \neq 0$. The matrix A is then non-singular.

Proof: It follows from the duality theorem of Kates ([12, Th.8, p.38]) that if $\mathbf{m} \in M_k$ and $\varphi_{2\mathbf{m}}$ is the spherical function on $(\mathcal{S}_{2k}, H)$ with $c(\varphi_{2\mathbf{m}}) = [c_1, \ldots, c_N]^\top$, that is $\varphi_{2\mathbf{m}} = \sum_{j=1}^{N} c_j \delta'_{C_j}$, then for any $\sigma \in \Omega$ the spherical polynomial $\Phi_{\mathbf{m}}$ on $\Omega = GL(n,\mathbb{R})/O(n)$ verifies

$$\Phi_{\mathbf{m}}(\sigma) = \sum_{j=1}^{N} c_j r_{\pi_j}(\sigma)(e, \ldots, e). \tag{5.34}$$

By Theorem 28, since the function $\varphi_{2\mathbf{m}}$ is a linear combination of the functions $\phi_v^{\pi_{2\mathbf{m}}}$, $v \in B_\pi$, we have $A\, c(\varphi_{2\mathbf{m}}) = e_{\pi_{2\mathbf{m}}} c(\varphi_{2\mathbf{m}})$. This means that for all $j = 1, \ldots, N$

$$\sum_{k=1}^{N} a_{jk} c_k = e_{\pi_{2\mathbf{m}}} c_j.$$

If we multiply this last equality by $r_{\pi_j}(\sigma)(e, \ldots, e)$ and sum over j, using the symmetry of the matrix A and (5.34), we obtain

$$\mathbb{E}\Phi_{\mathbf{m}}(S) = e_{\pi_{2\mathbf{m}}} \Phi_{\mathbf{m}}(\sigma).$$

On the other hand we know ([2]) that $\mathbb{E}\Phi_{\mathbf{m}}(S) = \frac{\Gamma_\Omega(p+\mathbf{m})}{\Gamma_\Omega(p)} \Phi_{\mathbf{m}}(\sigma)$. ∎

For any k the 1-dimensional trivial representation of $\mathcal{S}_{2k}$ corresponding to the partition $(2k)$ of $2k$ yields the constant eigenvector $[1, \ldots, 1]$ of the matrix A which corresponds to the eigenvalue $e_{\pi_{(2k)}} = \Gamma(p+k)/\Gamma(p) = p(p+1)\ldots(p+k-1)$ of A.

In order to construct other eigenvectors of A we can use Theorem 28 by constructing spherical vectors $u_{\pi_{2\mathbf{m}}}$ according to the method given in the proof of Theorem 29. Theorem 30 provides the corresponding eigenvalues of A.

Example. Consider $k=3$ and $\mathbf{m}=(2,1)\in M_3$. Then $2\mathbf{m}=(4,2)$. The subset $\mathcal{T}_o$ of $\mathcal{T}_{(4,2)}$ (we refer to the proof of Theorem 29) contains 3 tabloids

$$\overline{\underline{1\ \ 1^+2\ \ 2^+}}^{+}\ \overline{\underline{1\ \ 1^+3\ \ 3^+}}^{+}\ \overline{\underline{2\ \ 2^+3\ 3^+}}$$
$$\underline{3\ \ 3^+}\qquad\underline{2\ \ 2^+}\qquad\underline{1\ \ 1^+}$$

and the H-invariant function $\delta_{\mathcal{T}_o}$ is equal to

$$\delta_{\mathcal{T}_o}=\delta_{14}+\delta_{25}+\delta_{36}.$$

Its orthogonal projection on the Specht module $F_{(4,2)}$ is equal, up to a scalar factor $-\frac{1}{5}$, to

$$u=[1,1,-4,1,1,1,1,-4,1,1,1,-4,1,1,1]^\top$$

in the basis $\delta_{12},\delta_{13},\ldots,\delta_{56}$ of the space $\mathcal{A}(\mathcal{T}_{(4,2)})$ (this is very quick to find with XMaple, using the *GramSchmidt* function).

Note that the orthogonal decomposition of $\delta_{\mathcal{T}_o}$ is $\delta_{\mathcal{T}_o}=\frac{1}{5}[1,\ldots,1]-\frac{1}{5}u$; the $F_{(4,2)}^{\perp}$ component of $\delta_{\mathcal{T}_o}$ is a constant function. The coefficients of the vector u in the standard basis $\{e_{24},\ldots,e_{56}\}$ of $F_{(4,2)}$ are $[3,-3,1,1,1,-4,1,1,1]$ and do not present any special regularity.

Let us compute the eigenvector $c(\phi_{e_{24}}^{\pi_{(4,2)}})$ of the matrix A, corresponding to the eigenvalue $e_{\pi_{(4,2)}}=\frac{\Gamma_\Omega(p+(2,1))}{\Gamma_\Omega(p)}=p(p+1)(p-\frac{1}{2})$. Up to a factor $|H|$, the N coefficients of this vector are given by

$$(e_{t_{\pi_j}^{-1}\tau_{\epsilon(j)}\mathbf{t}},u),\quad j=1,\ldots,N$$

where $(\epsilon(j),\pi_j)$ are particular pairs and $\mathbf{t}=t_{24}$, the first standard tableau. We obtain an eigenvector

$$v_{24}=5\times[1,1,2,0,-1,-1,0,0,1,0,0,-1,-2,1,-1]^\top.$$

Similarly we compute

$$\begin{aligned}
v_{25}&=\ 5\times[-1,0,1,-1,-2,1,0,1,2,0,0,0,-1,1,-1]^\top\\
v_{26}&=\ 5\times[0,-1,1,1,-1,0,0,-1,1,0,0,1,-1,2,-2]^\top\\
v_{34}&=\ 5\times[1,1,2,-1,0,0,-1,-1,1,-2,-1,0,0,1,0]^\top\\
v_{35}&=\ 5\times[0,-1,1,0,0,1,-1,1,2,-1,-2,-1,0,1,0]^\top\\
v_{36}&=\ 5\times[-1,0,1,1,0,-1,-2,0,1,-1,-1,1,0,2,0]^\top\\
v_{45}&=\ 5\times[0,0,0,0,0,2,-2,1,1,-1,-1,1,1,-1,-1]^\top\\
v_{46}&=\ 5\times[0,0,0,1,-1,1,-1,0,-1,-1,-2,2,1,1,0]^\top\\
v_{56}&=\ 5\times[1,0,-1,2,1,0,-1,0,0,-2,-1,1,-1,1,0]^\top.
\end{aligned}$$

The 9 vectors $v_{24},\ldots,v_{56}$ are a basis of the eigenspace of A corresponding to the eigenvalue $p(p+1)(p-\frac{1}{2})$.

In order to find the eigenspace for the eigenvalue $e_{\pi_{(2,2,2)}}=\frac{\Gamma_\Omega(p+(1,1,1))}{\Gamma_\Omega(p)}=p(p-1)(p-\frac{1}{2})$ we could work with the representation $\pi_{(2,2,2)}$ of $\mathcal{S}_6$ but it is simpler to

determine the orthogonal complement of the linear space generated by the constant vector $l = [1, \ldots, 1]^\top$ and the vectors $v_{24}, \ldots, v_{56}$. An orthogonal basis of the eigenspace for the eigenvalue $p(p-1)(p-\frac{1}{2})$ is given by

$$\begin{aligned}
w_1 &= [\ 4,-2,-2,-2,-2,-2,-2,1,1,1,1,1,1,1,1]^\top \\
w_2 &= [\ 0,2,-2,0,0,0,0,-1,1,1,-1,-1,1,1,-1]^\top \\
w_3 &= [\ 0,0,0,2,-2,0,0,-1,1,-1,1,-1,1,-1,1]^\top \\
w_4 &= [\ 0,0,0,0,0,2,-2,-1,-1,1,1,-1,-1,1,1]^\top \\
w_5 &= [\ 0,0,0,0,0,0,0,1,-1,-1,1,-1,1,1,-1]^\top.
\end{aligned}$$

Now, in order to compute the matrix A, define the matrix P to be the matrix with column vectors $l, v_{24}, \ldots, v_{56}, w_1, \ldots, w_5$. Writing $e_1 = e_{\pi_{(6)}} = p(p+1)(p+2)$, $e_2 = e_{\pi_{(4,2)}} = p(p+1)(p-\frac{1}{2})$ and $e_3 = e_{\pi_{(2,2,2)}} = p(p-1)(p-\frac{1}{2})$ and defining $D = \mathrm{diag}(e_1, e_2, \ldots, e_2, e_3, \ldots, e_3)$ where e_2 appears 9 times and e_3 5 times, we have

$$A = P\, D\, P^{-1}.$$

Let us observe that if we sum up the coefficients of the eigenvectors of the matrix A indexed by particular pairs (ϵ, π) with the same $\pi \in \mathcal{S}_k$, we obtain the eigenvectors of the matrix B corresponding to the same eigenvalue. This is in fact the same contraction procedure that we use to derive the matrix B from the matrix A. For $k = 3$ we get in this way one eigenvector $[1,2,2,2,4,4]^\top$ for e_1, 4 independent eigenvectors $[1,3,-1,-1,1,-3]^\top, [1,-1,3,-1,-3,1]^\top, [1,3,-1,-1,-3,1]^\top, [-1,1,1,-3,-1,3]^\top$ for e_2 and one eigenvector $[4,-4,-4,-4,4,4]^\top$ for the eigenvalue e_3. This allows us to construct the matrix B as $B = R\,\mathrm{diag}(e_1, e_2, e_2, e_2, e_2, e_3)\, R^{-1}$ where R is the matrix with column vectors the eigenvectors given above.

Remark. Another method for the determination of the eigenvectors of the matrix A corresponding to the eigenvalue $e_{\pi_{2\mathbf{m}}} = \frac{\Gamma_\Omega(p+\mathbf{m})}{\Gamma_\Omega(p)}$, is the following. It is based on the knowledge of the spherical functions $\varphi_{2\mathbf{m}}$ on $(\mathcal{S}_{2k}, H)$ which are equal to the vectors of coefficients of zonal polynomials $\Phi_{\mathbf{m}}$, which are in turn accessible in tables.

If $\Phi_{\mathbf{m}}(\sigma) = \sum_{i=1}^{p(k)} d_i\, r_i(\sigma)(e, \ldots, e)$ then, by [12], $\varphi_{2\mathbf{m}} = \sum_{j=1}^{N} c_j \delta'_{C_j}$ with

$$c_j = d_i/n_i \quad \text{when } C_j \subset D_i$$

and n_i equal to the number of cosets $C_j = T'_j H$ contained in the double coset D_i.

For example, when $k = 3$ and $\mathbf{m} = (2,1)$ then the coefficients of $\Phi_{(2,1)}$ are 1,1,-2 and the numbers n_i are equal to $1, 6, 8$. It follows that

$$c(\varphi_{2\mathbf{m}}) = [1, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}, \frac{-2}{8}].$$

This is one eigenvector of the matrix A corresponding to the eigenvalue $e_2 = \frac{\Gamma_\Omega(p+(2,1))}{\Gamma_\Omega(p)}$. We should find 9 independent eigenvectors.

The irreducible representation space for the representation $\pi_{2\mathbf{m}}$ may be realised as the linear span of all $\mathcal{S}_{2k}$-translates of $\varphi_{2\mathbf{m}}$:

$$\varphi^g_{2\mathbf{m}} : \ x \to \varphi_{2\mathbf{m}}(gx), \quad g \in \mathcal{S}_{2k}$$

(in the preceding considerations it was realized as the Specht module $F_{2\mathbf{m}}$). As the functions $\varphi^g_{2\mathbf{m}}$ are right H-invariant, it would be sufficient to find $d_{\pi_{2\mathbf{m}}}$ permutations g such that the functions $\varphi^g_{2\mathbf{m}}$ are independent.

What we did using Theorem 28 was to find $d_{\pi_{2\mathbf{m}}}$ independent eigenvectors using Theorem 27 and the explicit construction of the standard basis of $F_{2\mathbf{m}}$. What we suggest in this Remark is that if we find $d_{\pi_{2\mathbf{m}}}$ permutations $g \in \mathcal{S}_{2k}$ such that the functions $\varphi^g_{2\mathbf{m}}$ are independent then we reach our objective, too.

Let us come back to the example $k = 3$ and $\mathbf{m} = (2,1)$. If we take $g = t_{(23)}$ then the functions $\varphi_{2\mathbf{m}}$ and $\varphi^g_{2\mathbf{m}}$ are independent since, up to the factor $|H|$ we have $\varphi_{2\mathbf{m}}(T'_1) = 1,\ \varphi_{2\mathbf{m}}(T'_2) = \frac{1}{6}$ and $\varphi^g_{2\mathbf{m}}(T'_1) = \frac{1}{6},\ \varphi^g_{2\mathbf{m}}(T'_2) = 1$. One may try to add more and more permutations to $\{id, t_{(23)}\}$ until 9 independent functions $\varphi^g_{2\mathbf{m}}$ are obtained. We do not think it is possible to find a systematic method of choosing the g's.

6 Appendix

Theorem 31 *Let Ω be the symmetric cone of positive definite real symmetric or complex hermitian matrices. Let V be the associated Jordan algebra. For $(A_1, ..., A_l)$ and $(A'_1, ..., A'_l)$ given in Ω^l, we have that*

$$\exists u \in K \quad A'_j = u.A_j,\ j = 1, ..., l \ \Leftrightarrow \ \forall m_1, ..., m_j \in \{1, ..., l\} \quad \mathrm{tr}(A_{m_1}...A_{m_j}) = \mathrm{tr}(A'_{m_1}...A'_{m_j})$$

Proof: $\Rightarrow$ is evident.

The $\Leftarrow$ direction follows from the unitary implementation theorem for finite dimensional von Neumann algebras (see [4], Thms. 5.5 and 5.6 or [10], Thm. 7.2.9), which also has a real orthogonal version ([11], Thm. 65). ∎

Corollary 32 *The algebra $I(V)^K_k$ of K-invariant polynomials in k variables on V is generated by the traces of the words in $h_1, \ldots, h_k \in V$.*

Theorem 33 *Let $\mathcal{E}_k$ be the space of k-multilinear forms on V spanned by the forms*

$$(h_1, ..., h_k) \in V^k \mapsto r_\pi(e)(h_1, ..., h_k), \quad \pi \in \mathcal{S}_k.$$

Then the space $\mathcal{E}_k$ is equal to the space $\mathcal{F}_k$ of K-invariant k-multilinear forms on V.

Remark. The K-invariance is defined as follows: denote $u.h = uhu^*$, for $h \in V$ and $u \in K$. Then f is in $\mathcal{F}_k$ if

$$f(u.h_1, ..., u.h_k) = f(h_1, ..., h_k).$$

Proof: Obviously $\mathcal{E}_k \subset \mathcal{F}_k$. In order to prove the reverse inclusion we remark that k-multilinear forms on V can be identified with homogeneous polynomials of degree k on V. The polynomials $r_\pi(e)$ are obviously K-invariant. We then use Corollary 32. ∎

Let us now come back to the space $\mathcal{P}_k$ defined in the Introduction.

Corollary 34 *The linear space $\mathcal{P}_k$ is generated by $(r_\pi)_{\pi\in\mathcal{S}_k}$, where*

$$r_\pi(x, h_1, \ldots, h_k) = r_\pi(x)(h_1, \ldots, h_k).$$

Proof: By Theorem 33 it is sufficient to show that $\mathcal{P}_k$ is isomorphic to $\mathcal{F}_k$ via the map

$$q(h_1, \ldots, h_k) := F(e, h_1, \ldots, h_k), \quad F \in \mathcal{P}_k,$$

the inverse of which is given by

$$F(x, h_1, \ldots, h_k) := q(x^{1/2}h_1x^{1/2}, \ldots, x^{1/2}h_kx^{1/2}), \quad q \in \mathcal{F}_k, x \in \Omega. \tag{6.35}$$

The proof goes along the same lines as the proof of Proposition XIV.1.1 in [2], which corresponds to our Corollary for $k = 1$. The only substantial difference is that we have to show that F defined in (6.35) is a polynomial with respect to x. This follows from Corollary 32 applied to $q \in \mathcal{F}_k$. ∎

We finish with the formal definition of the Wishart expectation operators T_p. We prove, directly, the commutativity of T_p's for different p's. This is more general than Corollary 26 because $(r_\pi)_{\pi\in\mathcal{S}_k}$ may not be independent (they always are a generating family for $\mathcal{P}_k$). When they are independent, Corollary 26 implies Theorem 36 (the matrices $A(p)$ are the matrices representative of the operators T_p in the basis $(r_\pi)_{\pi\in\mathcal{S}_k}$).

Definition 35 (of "Wishart expectation" operators).
Let T_p be an operator on $\mathcal{P}_k$ defined by

$$T_pF(\sigma; h_1, ..., h_k) = I\!EF(W_{p,\sigma}, h_1, ..., h_k)$$

where $W_{p,\sigma}$ is a random variable with Wishart distribution with parameters p, σ.

Theorem 36 *If p and p' are in Λ, the Gindikin set, then the operators T_p and $T_{p'}$ commute.*

Proof: Let $f(x) = f_\theta(x) = e^{\langle\theta,x\rangle}$. It is sufficient to show that for all $\theta \in -\Omega$

$$T_{p'} \circ T_p(f)(\sigma) = T_p \circ T_{p'}(f)(\sigma)$$

(we omit $h_1, ..., h_k$ in the notation). We have

$$T_{p'} \circ T_p(f)(\sigma) = \int_\Omega \int_\Omega e^{\langle\theta,x\rangle}\gamma_{p',y}(dx)\gamma_{p,\sigma}(dy).$$

We first consider $\sigma = e$ and use $y = P(y^{1/2})(e)$. Knowing that $g.\gamma_{p,\sigma} = \gamma_{p,g.\sigma}$ we obtain

$$\begin{aligned} T_{p'} \circ T_p(f)(e) &= \int\int e^{\langle\theta,x\rangle}\gamma_{p',P(y^{1/2})(e)}(dx)\gamma_{p,e}(dy) \\ &= \int\int e^{\langle\theta,P(y^{1/2})x\rangle}\gamma_{p',e}(dx)\gamma_{p,e}(dy). \end{aligned}$$

It is well known that if X and Y are two independent K-invariant random variables then the laws of $P(X^{1/2})Y$ and $P(Y^{1/2})X$ are identical. Since the measures $\gamma_{p',e}$ and $\gamma_{p,e}$ are K-invariant, we can apply this result to our last formula and it follows that

$$T_{p'} \circ T_p(f)(e) = T_p \circ T_{p'}(f)(e). \tag{6.36}$$

In the general case

$$\begin{aligned} T_{p'} \circ T_p(f)(\sigma) &= \int\int e^{\langle \theta, x \rangle} \gamma_{p',y}(dx) \gamma_{p,P(\sigma^{1/2})(e)}(dy) \\ &= \int\int e^{\langle \theta, x \rangle} \gamma_{p',P(\sigma^{1/2})y}(dx) \gamma_{p,e}(dy) = \int\int e^{\langle \theta, P(\sigma^{1/2})x \rangle} \gamma_{p',y}(dx) \gamma_{p,e}(dy) \\ &= T_{p'} \circ T_p(f_{P(\sigma^{1/2})\theta})(e), \end{aligned}$$

which, using (6.36), yields the desired result. ∎

7 Acknowledgments

We would like to thank Jacques Faraut for helpful comments on the usefulness of representation theory and the work of Kates for our problem. We would also like to thank Roland Speicher for comments on Specht's theorem and von Neumann algebras. We are grateful to Patrice Sawyer for helping us draw figures in our paper.

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